A Short Intrduction to Representable Functors

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Representable functors are a particular class of presheaves and here we try to consider why representable functors are important in algebraic geometry. Our viewpoint is closed to [Gro74] and [DeG80] and readers can find details there. We start our introduction to representable functors with a question. Is the categorical structure of varieties unique? As is well known, in classical algebraic geometry, we use regular maps to define the category of varieties. Here, we restrict ourselves to affine varieties for simplicity.

We assume k is an algebraically closed field, and any polynomial $f \in k[x_1, ..., x_n]$ defines a function $k^n \to k$ by $(a_1, ..., a_n) \mapsto f(a_1, ..., a_n)$. Given a subset $I \subseteq k[x_1, ..., x_n]$, we correspond it with an *algebraic subset* (or *affine variety*) of k^n, $Z(I) = \{(a_1, ..., a_n \in k^n)|f(a_1, ..., a_n), \forall f \in I\}$. The topology on k^n with all algebraic subsets as closed subset is called the Zariski topology. It's easy to see algebraic subsets satisfy axioms of closed subsets and it actually forms a topology.

Conversely, given any subset $X \subseteq k^n$, we can associate it with an ideal $I(X) = \{f \in k[x_1, ..., x_n]|f(a_1, ..., a_n) = 0, \forall(a_1, ..., a_n) \in X\}$ of $k[x_1, ..., x_n]$. If $g \in \mathrm{rad}(I(X))$, then $g^m \in I(X) \Rightarrow g(a_1, ..., a_n)^m = 0 \Rightarrow g(a_1, ..., a_n) = 0$. Hence $g \in I(X)$. This fact implies $I(X)$ is an radical ideal. The Hilbert's Nullstellensatz[1] tells us that for any ideal $J \subseteq k(x_1, ..., x_n)$, we will have $I(Z(J)) = \mathrm{rad}(J)$. Then there is a bijection between the collection of algebraic subsets of k^n and the set of radical ideals of $k[x_1, ..., x_n]$. Hence, Hilbert's Nullstellensatz can transfer geometric information into algebraic information. Also from this theorem, we have the following theorem:

Theorem 1. *If k is an algebraically closed field and A is a k-algebra of the form $A = k[x_1, ..., x_n]/I(X)$, for some algebraic subset X iff A is reduced and finitely generated as a k-algebra. These k-algebras are often called affine k-algebras.*

Proof. "$\Rightarrow$". It's obvious for $I(X)$ is radical.

"$\Leftarrow$". A will have the form $A = k[x_1, ..., x_n]/J$ where J is radical. Then $I(Z(J)) = J$. □

For any algebraic subset X we can associate it with an affine k-algebra $k[x_1, ..., x_n]/I(X)$. We know affine k-algebras form a full subcategory of the category of k-algebras. To make such correspondence above a functor, we need define the category of algebraic

[1] If k is any field, A is a k-algebra of finite type, then its Jacobson radical ideal coincides with its nilradical ideal, $j(A) = \mathrm{rad}(A)$. Only if moreover k is algebraically closed, $I(Z(J)) = \mathrm{rad}(J)$ will be true. See [Bos13] Section 3.2 for more detail.

subsets first. Naturally, we may take polynomials as morphisms between algebraic subsets, because we deal with polynomial rings here. These morphisms will be called *regular maps*. For two algebraic subsets $X \subseteq k^n$, $Y \subseteq k^m$, a regular map is such that $F : X \to Y, (a_1, ..., a_n) \mapsto (f_1(a), ..., f_m(a))$, where $f_i \in k[x_1, ..., x_n]$. If $g = (g_1, .., g_m)$ is another regular map, then $g = f$ iff $g_i = f_i \Leftrightarrow g_i - f_i \in I(X)$. Hence, we can choose $f_i \in k[x_1, ..., x_n]/I(X)$. A classical well known result tells us that:

Theorem 2. *The category of algebraic subsets with regular maps as morphisms is equivalent to the category of affine k-algebras with arrows reversed, where k is algebraically closed.*

But to make affine varieties form a category, regular maps are not the only choice. Equipped with Zariski topology, k^n is actually a topological space. Hence, any algebraic subset will naturally have the topology induced from k^n. And we can let morphisms between affine varieties be continuous maps. This also defines a category, but it will not have the functorial property as in Theorem 2. And worse, it won't have the polynomial algebraic structure we need. Hence, there exists a problem how to make the categorical structure of varieties with the algebraic information of polynomials unique.

Here, we embed the category of varieties into a larger category and let morphisms between affine varieties meet the restrictions in the larger category, so that the regular map can become unique. And the larger category is the category of presheaves.

Now we define an affine variety is a functor called the *variety functor* $V : \mathbf{k-alg} \to \mathbf{Sets}$, where k is an arbitrary commutative ring with a unit. This variety functor is determined by a series of polynomials $f_1, ..., f_r \in k[x_1, ..., x_m]$. For any k-algebra k', we define $V(k') = \{x = (a_1, ..., a_m) \in k'^m | f_i(a) = 0, i = 1, ..., r\}$. If $h : k' \to k''$ is any morphism in the category of k-algebras, $V(h)(a_1, ..., a_m) = (h(a_1), ..., h(a_m))$. This is well defined, because $f_i(h(a) = h(f_i(a)) = 0$. The variety functor is actually a functor.

Then we consider $\Gamma(V) = k[x_1, ..., x_m]/(f_1, ..., f_r)$, we will have $V(k') \xrightarrow{\sim} \mathrm{Hom}_k(\Gamma(V), k')$. We prove it as follows:

Proof. Given an element $a = (a_1, ..., a_m) \in V(k')$, we define a morphism $\sigma_a : \Gamma(V) \to k'$ such that $\sigma_a(g) = g(a_1, ..., a_m), g \in \Gamma(V)$. σ_a is actually a morphism in the category of k-algebras. Conversely, for any morphism $\sigma : \Gamma(V) \to k'$, we let $a_i = \sigma(\bar{x}_i)$, where $\bar{x}_i$ is the image of x_i in $\Gamma(V)$. Then $(a_1, ..., a_m) \in V(k')$. The statement above actually defines a bijection between $V(k')$ and $\mathrm{Hom}_k(\Gamma(V), k')$.

If $h : k' \to k''$ is any morphism in the category of k-algebras, we will have the following commutative diagram:

$$\begin{array}{ccc} V(k') & \xrightarrow{\sim} & \mathrm{Hom}_k(\Gamma(V), k') \\ {\scriptstyle V(h)}\downarrow & & \downarrow{\scriptstyle h_*} \\ V(k'') & \xrightarrow{\sim} & \mathrm{Hom}_k(\Gamma(V), k'') \end{array}$$

If $a \in V(k')$, $h(a)$ defines a morphism $\Gamma(V) \to k''$ such that $g \mapsto g(h(a)) = g(h(a_1), ..., h(a_m)) = h(g(a))$. Hence the bijection above is functorial. □

Now we have a natural isomorphism $V(-) \xrightarrow{\sim} \mathrm{Hom}_k(\Gamma(V), -)$, and the variety functor $V(-)$ can be identified with the representable functor $\mathrm{Hom}_k(\Gamma(V), -)$.

We consider a new variety functor W defined by $g_1, ..., g_l \in k[t_1, ..., t_n]$. A *regular map* is a natural transformation $\varphi : V \to W$ such that $\varphi = (\varphi_1, ..., \varphi_n)$ where $\varphi_i \in \Gamma(V) = k[x_1, ..., x_m]/(f_1, .., f_r)$. This regular map is obviously natural. But are all natural transformations between variety functors regular? The answer is positive.

If $\varphi : V \to W$ is any transformation between variety functors, we will have a natural transformation $\varphi : \mathrm{Hom}_k(\Gamma(V), -) \to \mathrm{Hom}_k(\Gamma(W), -)$. The Yoneda's lemma tells us that, the transformation is induced by a morphism $\psi : \Gamma(W) \to \Gamma(V)$. We let $\varphi_i = \psi(\bar{t_i}), i = 1, ..., n$, and then $\varphi = (\varphi_1, ..., \varphi_n)$. To prove this equality, $a = (a_1, ..., a_m) \in V(k')$ corresponds to $\sigma_a : \Gamma(V) \to k'$, then $\varphi(\sigma_a) = \sigma_a \circ \psi$, which corresponds to $y_i = \sigma_a \circ \psi(\bar{t_i}) = \sigma_a(\varphi_i) = \varphi_i(a)$. The natural transformation between variety functors must be regular maps.

In fact, from the natural isomorphism $V(-) \xrightarrow{\sim} \mathrm{Hom}_k(\Gamma(V), -)$ and Yoneda's lemma, we have the following corallay:

Corollary 3. *The category of variety functors is equivalent to the category of k-algebras of finite presentation, where k is an arbitrary commutative ring with a unit.*

Now we move on to schemes. If X is an arbitrary scheme, we can identify it with its representable functor $h_X = \mathrm{Hom}_{\mathbf{Sch}}(-, X) : \mathbf{Sch}^{op} \to \mathbf{Sets}$ via Yoneda's lemma. Generally speaking, there are four types of coverings $(\mathrm{fpqc}) \geq (\mathrm{fppf}) \geq (\mathrm{et}) \geq (\mathrm{zar})$ in the category of schemes. Equipped with any one of them, the representable functor h_X will be a sheaf. This theorem will be proved when we talk about Grothendieck topology but here we only need the theorem on Zariski site.

Theorem 4. *If X, Y are two arbitrary schemes and $\cup_i X_i = X$ is an open covering of X, then the following diagram is an equalizer:*

$$\mathrm{Hom}_{\mathbf{Sch}}(X, Y) \longrightarrow \prod_i \mathrm{Hom}_{\mathbf{Sch}}(X_i, Y) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathbf{Sch}}(X_i \cap X_j, Y) \tag{1}$$

Proof. Given a set of morphisms $f_i : X_i \to Y$ satisfying the condition $f_i|X_i \cap X_j = f_j|X_i \cap X_j$. At the level of underlying topological space, it's obvious to see there will exist a unique morphism $f : |X| \to |Y|$. Then we should prove the part of sheaves.

For any open subset $U \subseteq Y$:

$$\begin{array}{ccccc} \mathcal{O}_X(f^{-1}(U)) & \longrightarrow & \prod_i \mathcal{O}_X(f^{-1}(U) \cap X_i) & \rightrightarrows & \mathcal{O}_X(f^{-1}(U) \cap X_i \cap X_j) \\ & \nwarrow^{\exists !} & \uparrow{\scriptstyle \prod_i f_i^\sharp} & & \\ & & \mathcal{O}_Y(U) & & \end{array}$$

The morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is induced by the universal property of the equalizer. Hence it will satisfy the compatible conditions. □

Corollary 5. *If Y is an affine scheme $\mathrm{Spec}\ R$, and X is an arbitrary scheme, then we will have the natural isomorphism $\mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}\ R) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Rings}}(R, \mathcal{O}_X(X)), f \mapsto f^\sharp$.*

Proof. If X is also an affine scheme, then the corollary is a standard theorem in any textbook of algebraic geometry. We cover X by a set of affine open subschemes $X = \cup_i X_i$, and cover $X_i \cap X_j$ by $X_i \cap X_j = \cup_k X_{ijk}$ wher X_{ijk} is affine open in both X_i and X_j. Assume $\mathcal{O}_X(X) = A, \mathcal{O}_X(X_i) = A_i, \mathcal{O}_X(X_{ijk}) = A_{ijk}$. The sheaf property of $\mathcal{O}_X$ tells us that there is an equalizer

$$A \longrightarrow \prod_i A_i \rightrightarrows \prod_{ijk} A_{ijk}$$

The representable functor $\mathrm{Hom}(R, -)$ preserves limits. Hence we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}\ R) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathbf{Sch}}(X_i, \mathrm{Spec}\ R) & \rightrightarrows & \prod_{i,j,k} \mathrm{Hom}_{\mathbf{Sch}}(X_{ijk}, \mathrm{Spec}\ R) \\ \downarrow & & \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathbf{Rings}}(R, A) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathbf{Rings}}(R, A_i) & \rightrightarrows & \prod_{i,j,k} \mathrm{Hom}_{\mathbf{Rings}}(R, A_{ijk}) \end{array}$$

The dotted arrow will also be an isomorphism by diagram chasing. □

According to the Theorem 4, $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ can be obtained from $\mathrm{Hom}_{\mathbf{Sch}}(X_i, Y)$ where X_i is affine open. Hence, if we restrict ourselves to the category of affine schemes, the information of representable functors won't be lost, and we can view the functor $h_Y$as $\mathbf{Aff}^{op} \to \mathbf{Sets}$. But the category of affine schemes is equivalent to the category of rings. It's equivalent to say h_Y is functor such that $\mathbf{Rings} \to \mathbf{Sets}, h_Y(A) = \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\ A, Y)$.

If we consider the category of schemes over a base affine scheme $\mathrm{Spec}\ R$, then h_X : $\mathbf{R} - \mathbf{alg} \to \mathbf{Sets}$. And we can conclude that via $h : \mathbf{Sch}/R \to \mathbf{Sets}^{\mathbf{R-alg}}$, the category of schemes over R is equivalent to a full subcategory of $\mathbf{Sets}^{\mathbf{R-alg}}$.

Yoneda's lemma plays a central role in the discussion above. It helps us establish an equivalence between geometric objects and functors to some degree. Generally speaking there are three advantages to view a scheme as a functor.

(1). Compared with describing the point structure of a scheme X directly, to describe the structure of Y-valued points of X, that it, $\mathrm{Hom}_{\mathbf{Sch}}(Y, X)$ is much easier.

For example, we can consider the projective scheme $\mathbb{P}_k^n$ where k is a field. According to the standard teaching contents on projective schemes, we know $\mathbb{P}_k^n$ is glued by $(n+1)$'s affine spaces $\mathbb{A}_k^n$. But it's not affine in general. And there is another viewpoint to describe the point structure of projective schemes using homogeneous prime ideal. For a graded polynomial ring $k[X] = k[x_0, ..., x_n]$, $\mathbb{P}_k^n = \mathrm{Proj}\ k[X]$ can be locally identified with an affine scheme, which means for any $D_+(f) \subseteq \mathbb{P}_k^n$, $D_+(f) \cong \mathrm{Spec}\ k[x_0, ..., x_n]_{(f)}$.

If $k \to k'$ is a field extension, we now try to describe the structure of $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\ k', \mathbb{P}_k^n)$, which is in fact the standard projective space of k'. $\mathbb{P}_k^n$ is glued by affine schemes $X_i = \mathrm{Spec}\ k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$. For any k'-valued point a : $\mathrm{Spec}\ k' \to \mathbb{P}_k^n$, if it factors through some X_i, we will obtain a morphism $\sigma_a : A_i = k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}] \to k'$, and $(\sigma_a(\frac{x_o}{x_n}) : ... : \sigma_a(\frac{x_n}{x_n}))$ in the projective space of k'. This is well defined and independent from the choice of X_i.

Conversely, we assume $(a_0 : ... : a_n)$ is in the projective space of k', and then we have a representation $\tau : k[x_0, ..., x_n] \to k'$ of it, such that $\tau(x_i) = a_i$. If $a_i \neq 0$, we will have

$\sigma_i : A_i \to k'$ satisfying $\sigma_i(\frac{x_j}{x_i}) = a_j a_i^{-1}$, which is independent from the choice of a_i. Then $a : \mathrm{Spec}\ k' \to \mathrm{Spec}\ A_i \hookrightarrow \mathbb{P}^n_k$.

Therefore for the projective scheme $\mathbb{P}^n_k$, $\mathbb{P}^n_k(k') = \mathrm{Hom}_k(\mathrm{Spec}\ k', \mathbb{P}^n_k)$ is just the projective space over k' in the usual sense. Although for any other more general k-algebra R, $\mathbb{P}^n_k(R)$ may not be such a projective space, there always exists an explicit description of this R-valued points set, even for any scheme X over k.

$$\mathbb{P}^n_k(X) = \{p : \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L}\}/\sim$$

where $\mathcal{L}$ is a *line bundle* and $p \sim p'$ if there exists an isomorphism $\varphi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ satisfying $\varphi \circ p = p'$.

Better examples are *Hilbert schemes* and *Grassmannians*.

(2). Compared with constructing a scheme directly, constructing a functor is much easier. But then the problem will be how to prove this functor is representable. There many examples, such as Hilbert schemes, Picard schemes and quotient schemes all of which are due to Grothendieck's work in [FGA]. We give the example of Hilbert schemes here.

Suppose S is a locally Noetherian scheme, $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times S$. Consider the closed subscheme $Y \subseteq \mathbb{P}^n_S$ such that Y is flat over S and the set $\mathcal{H}ilb(S)$ consists of these closed subschemes Y. For any morphism $f : T \to S$, there a is a natural map $\mathbb{P}^n_T = \mathbb{P}^n_{\mathbb{Z}} \times T \xrightarrow{\mathrm{id}\times f} \mathbb{P}^n_{\mathbb{Z}} \times S = \mathbb{P}^n_S$. Then we will have the following pullback diagram:

$$\begin{array}{ccc} Y' & \hookrightarrow & \mathbb{P}^n_T \\ \downarrow & & \downarrow \mathrm{id}\times f \\ Y & \hookrightarrow & \mathbb{P}^n_S \end{array}$$

then we will have a morphism $\mathcal{H}ilb(f) : \mathcal{H}ilb(S) \to \mathcal{H}ilb(T), Y \mapsto Y'$. In [FGA], Grothendieck proves that this Hilbert functor is represented by a locally Noetherian scheme $Hilb_{\mathbb{P}^n}$ such that $\mathcal{H}ilb(-) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sch}}(-, Hilb_{\mathbb{P}^n})$. In fact, it's very hard to describe the points in $Hilb_{\mathbb{P}^n}$ directly, but the valued points of Hilbert functor are easier to describe.

Sometimes, the functor we construct may not be representable and this motivates more abstract concepts like *algebraic spaces* and *algebraic stacks*. According to the Theorem 4, to prove a functor representable, we should at least prove it's a sheaf, but the set of presheaves is much bigger than the set of sheaves. If a functor constructed from a moduli problem can be represented by a scheme X, then X will be called the *fine moduli space* of this functor (or moduli problem). If it's not representable, there are two methods to deal with this problem. The first is to embed the category of schemes into a larger category, and then the functor constructed may be represented in this new larger category. The larger category can be the category of algebraic spaces or algebraic stacks.

The second method is to find a representable functor close enough to the functor we are interested in. The underlying scheme of such representable functor will be called a *coarse moduli space*. More precisely, if $\mathcal{M}$ is a moduli functor we are interested in and M is the coarse moduli space of it with $\varphi : \mathcal{M} \to \mathrm{Hom}_{\mathbf{Sch}}(-, M)$, then (M, φ) should at least satisfy that for any representable functor $\mathrm{Hom}_{\mathbf{Sch}}(-, N)$ with a morphism $f : \mathcal{M} \to \mathrm{Hom}_{\mathbf{Sch}}(-, N)$, there exists a unique morphism $g : M \to N$ such that $g_* \circ \varphi = f$.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\varphi\quad} & \mathrm{Hom}_{\mathbf{Sch}}(-, M) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \exists! g_*} & \\ \mathrm{Hom}_{\mathbf{Sch}}(-, N) & & \end{array}$$

If the coarse moduli space exists, then it will be unique.

(3). The structure of the category of contravariant funcotrs (or presheaves) is similar to that of $\mathbf{Sets}$. In fact the category $\mathbf{Sets}^{\mathcal{C}^{op}}$ is an elementary topos, which means there exist colimits, limits, the subobject classifier and exponentials. Hence, many constructions which can be done in $\mathbf{Sets}$ can also be done in $\mathbf{Sets}^{\mathcal{C}^{op}}$. But this presheaf category is too big to find useful information, though all of the objects in it can be represented as a colimits of representable functors. Therefore, we should find a smaller category with enough information we want. The discussion above tells us that we can restrict the representable functor h_X to $\mathbf{R-alg} \to \mathbf{Sets}$. Here, it actually means the Yoneda embedding $h : \mathbf{Sch} \to \mathbf{Sets}^{\mathbf{Sch}^{op}}$ can be restricted to $h : \mathbf{Sch} \to \mathbf{Sets}^{\mathbf{R-alg}}$, where the codomain of the latter is much smaller than that of the former with information not lost.

All these motivate us to develop *functorial algebraic geometry* which means here functors are the first concept and spaces are the second concept. In the chapter of simplicial sets we know every simplicial set $X : \Delta^{op} \to \mathbf{Sets}$ has a CW-complex as its geometric realization. It's similar here. We can define *affine schemes* to be representable functors $\mathbf{Rings} \to \mathbf{Sets}$ and every affine scheme has associated *locally ringed space*. The standard construction is in any textbook about algebraic geometry. The category of locally ringed space is denoted by $\mathbf{LoRsp}$. Note that $\mathbf{LoRsp}$ is cocomplete since gluing process in it is valid and we have the spectral functor $\mathrm{Spec} : \mathbf{Rings}^{op} \to \mathbf{LoRsp}$. For any locally ringed space X:

$$\mathbf{Rings}^{op} \xrightarrow{\mathrm{Spec}} \mathbf{Rsp} \xrightarrow{\mathrm{Hom}_{\mathbf{LoRsp}}(-,X)} \mathbf{Sets}$$

then we haved defined a functor $\mathrm{Spec}^* : \mathbf{LoRsp} \to \mathbf{Sets}^{\mathbf{Rings}},\ X \mapsto \mathrm{Hom}_{\mathbf{LoRsp}}(-, X) \circ \mathrm{Spec}$ and it has a left adjoint functor

$$\mathrm{Spec}_! : \mathbf{Sets}^{\mathbf{Rings}} \to \mathbf{LoRsp},\ X \mapsto \operatorname*{colim}_{\mathrm{Hom}_{\mathbf{Rings}}(R,-) \to X} \mathrm{Spec}\, R$$

This is the associated geometric space for a functor $X : \mathbf{Rings} \to \mathbf{Sets}$, which is also denoted by $|X|$.

According to the adjunction

$$\mathrm{Spec}_! : \mathbf{Sets}^{\mathbf{Rings}} \rightleftarrows \mathbf{LoRsp} : \mathrm{Spec}^*$$

a functor $X : \mathbf{Rings} \to \mathbf{Sets}$ is a scheme which means $|X|$ is scheme in the classical sense iff adjunctions $X \to \mathrm{Spec}^*(|X|)$ and $|\mathrm{Spec}^*(|X|)| \to |X|$ are isomorphisms.[2] This is the *comparison theorem*. We won't talk more about these since our task in this section is not to establish the foundation for functorial algebraic geometry, but to just sketch motivations.

References

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[2]See the first chapter of [DeG80] for details.