Homotopical Algebra and Derived Categories

Yining CHEN

Shandong University
Email: yining_chen@mail.sdu.edu.cn

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Abstract

This notes is about homotopy theory of topological spaces and the localization of categories. We develop such a homotopy theory using techniques similar to simplicial sets. As for localization, we introduce two methods to construct the localization category. One is the model category developed first by Quillen and the other is the calculus of fractions which is in fact earlier than model categories. And we mainly focus on the example of chain complexes.

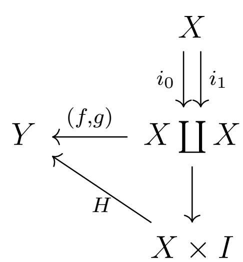
Contents

1 Homotopy Theory of Topological Spaces

1.1 Basic Concepts

There are two important notions in algebraic topology. The one is homology and the other is homotopy. Both of them are functors in some sense from some categories containing topological or geometrical information to algebraic categories such as the category of abelian groups $\mathbf{Ab}$. In this section we mainly introduce some basic facts about homotopy theory and a homotopy is actually a deformation of two functions.

Definition 1.1. *Given two continuous maps $f, g : X \to Y$ between topological spaces, a homotopy between f and g is a continuous map $H : X \times I \to Y$ such that $H|I \times 0 = f,\ H|I \times 1 = g$. The relation is written as $H : f \simeq g$. Or we can say there is the following commutative diagram:*



Fact 1.2. The homotopy relation in $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ is an equivalence relation. For a map $f : X \to Y$ we can define $H : X \times I \to Y,\ (x, t) \mapsto x$ then $f \simeq f$. If we already have $H : f \simeq g$, we let $K : X \times I \xrightarrow{(1_X, t \mapsto 1-t)} X \times I \xrightarrow{H} Y$, then $g \simeq f$. Given $H : f \simeq g,\ K : g \simeq h$, we have the following homotopy

$$K * H = \begin{cases} H(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ K(x, 2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

here $K * H : f \simeq h$. Then there is a quotien category of $\mathbf{Top}$ denoted by $\mathbf{HoTop}$ such that it has the same objects as $\mathbf{Top}$ and $\mathrm{Hom}_{\mathbf{HoTop}}(X, Y) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)/\sim$ is the homotopy class of continuous maps. We need to prove it's actually a category. For any homotopy classes $[f], [g]$ we define $[g] \circ [f] = [g \circ f]$ and this definition is independent from the choice of f and g. If we have $H : f_0 \simeq f_1,\ K : g_0 \simeq g_1$ we can define the homotopy $g_0 \circ f_0 \simeq g_1 \circ f_1$ as follows

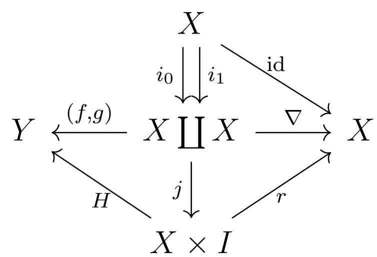
$$K \circ (H \times 1_I) : X \times I \xrightarrow{H \times 1_I} Y \times I \xrightarrow{K} Z$$

There is a natural functor $\gamma : \mathbf{Top} \to \mathbf{HoTop},\ X \mapsto X,\ f \mapsto [f]$. A *homotopy equivalence* is an isomorphism in $\mathbf{HoTop}$ i.e. if $f : X \to Y$ is a homotopy equivalence then there is a map $g : Y \to X$ such that $f \circ g \simeq 1_Y,\ g \circ f \simeq 1_X$. The class of homotopy equivalences is denoted by $\mathcal{H}$.

There is a universal property for $\mathbf{HoTop}$.

Theorem 1.3. $\mathbf{HoTop} \cong \mathbf{Top}[\mathcal{H}^{-1}]$

Proof. This is an equivalence between categories and you can see Definition 2.1 for the meaning of $\mathbf{Top}[\mathcal{H}^{-1}]$. Given a functor $F : \mathbf{Top} \to \mathcal{D}$ taking homotopy equivalences to isomorphisms, we only need to prove it preserves homotopy which means if $f \simeq g$ then $F(f) = F(g)$.



where $\nabla = (\mathrm{id}_X, \mathrm{id}_X)$ and $r(x,t) = x$. We prove r is a homotopy equivalence. If it's ture then $F(r)$ is an isomorphism and $F(j_0) = F(j_1) = F(r)^{-1}$. Then $F(f) = F(H \circ j_0) = F(H \circ j_1) = F(g)$.

Note $r \circ j_0 = r \circ j_1 = \mathrm{id}_X$. $j_0 \circ r : X \times I \to X \times I,\ (x,t) \mapsto (x,0)$ Define $X \times I \times I \to X \times I,\ (x,t,s) \mapsto (x,st)$. Then $j_o \circ r \simeq \mathrm{id}_{X\times I}$. Hence $[r]$ is an isomorphism with inverse $[j_0]$. □

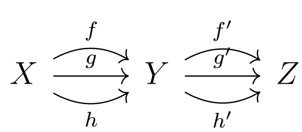
In **Top**, is the initial object and $*$ the one point space is the terminal object. The category of spaces with a point $\mathbf{Top}_*$ is actually the category $*/\mathbf{Top}$. In general we can define the relative category of topological spaces $\mathbf{Top}_{(2)}$ consisting of objects the pair (X, A) where $A \subseteq X$ and morphisms $f : (X, A) \to (Y, B)$ such that $f(A) \subseteq B$. And a homotopy in $\mathbf{Top}_{(2)}$ is actually a map $H : (X \times I, A \times I) \to (Y, B)$. Moreover if $f|A = g|A$ a homotopy relative to A written as $H : f \simeq g,\ \mathrm{rel}\ A$ is a homotopy $H : (X \times I, A \times I) \to (Y, B)$ such that $H(a,t) = f(a) = g(a)$ for all $a \in A,\ t \in I$. You can prove it's an equivalence relation between all maps satisfying $f|A = g|A$.

In $\mathbf{Top}_*$ the pointed homotopy $H : (X \times I, * \times I) \to (Y, *)$ is actually a homotopy relative to $*$.

Remark 1.4. With definitions above we can get a 2-category from **Top** i.e. a category enriched in **Cat** where the mapping category $\mathrm{Map}(X, Y)$ consists of continuous maps as objects and homotopy classes relative to $X \times \partial I$ as morphisms. This means $f, g : X \to Y$ are objects and $[F]$ where $F : X \times I \to Y, f \simeq g$ is morphism between f and g. Note $F \sim G$ if there is a map $H : X \times I \times I \to Y, F \simeq G,\ \mathrm{rel}\ X \times \partial I$ where $F|X \times \partial I = G|X \times \partial I = (f, g)$. Given morphisms $F_1 : f \simeq g,\ F_2 : g \simeq h$ let $F_2 * F_1$ be defined as in Fact 1.2. Then it's trivial to prove in the category $\mathrm{Map}(X, Y)$ axioms of categories are valid. Now let us define the composition

$$\circ : \mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$$

Next we need to prove this composition functor preserves compositions in the category of $\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y)$ which is just the axiom of 2-categories.



Given homotopies $F_1 : f \simeq g,\ F_2 : g \simeq h$ and $F_1' : f' \simeq g',\ F_2' : g' \simeq h'$, we need to prove the following diagram is commutative

$$\begin{array}{ccc} f' \circ f & & \\ {\scriptstyle F_1' \circ F_1} \Downarrow & \overset{(F_2' * F_1') \circ (F_2 * F_1)}{\searrow} & \\ g' \circ g & \underset{F_2' \circ F_2}{\Longrightarrow} & h' \circ h \end{array}$$

which is just to prove the interchange law $(F_2' \circ F_2) * (F_1' \circ F_1) = (F_2' * F_1') \circ (F_2 * F_1)$ is true. Note here $F_1' \circ F_1$ is defined as in Fact 1.2 to be $F_1' \circ (F_1 \times \mathrm{id}_X)$. It can be easily proved that

$$(F_2' \circ F_2) * (F_1' \circ F_1) = (F_2' * F_1') \circ (F_2 * F_1) = \begin{cases} F_1'(F_1(x, 2t), 2t),\ 0 \leq t \leq \frac{1}{2} \\ F_2'(F_2(x, 2t-1), 2t-1),\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

Next we introduce the concept of homotopy groups especially the fundamental groupoids. Here we use the method similar to that in simplicial homotopy theory.

1.2 Homotopy Groups

We define the **topological standard n-simplex** to be $|\Delta^n| = \{(t_0, ..., t_n) | t_i \geq 0, i = 0, ..., n,\ \sum_{i=0}^{n} t_i = 1\}$. Then the 0-simplex $|\Delta^0|$ is just a point and $|\Delta^1|$ is an interval. $|\partial \Delta^n|$ denotes the boundary of $|\Delta^n|$ and $|\Lambda_i^n|$ is obtained by throwing the i-th face of the boundary. For $|\Delta^n|$, the i-th face is the subspace of it where $t_i = 0$. A 0-simplex is a point and a 1-simplex is actually a path. If $\sigma : |\Delta^1| \to X$ is a 1-simplex of X, then it has a natural orientation from $x = d_1(\sigma)$ to $y = d_0(\sigma)$. In algebraic topology, $\pi_0(X)$ is defined to be the set of path components of X. But, we can also express this fact in categorical language:

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^1|, X) \overset{d_1}{\underset{d_0}{\rightrightarrows}} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^0|, X) \longrightarrow \pi_0(X) \tag{1}$$

$\pi_0(X)$ is translated as a coequalizer in **Sets**. For any map $f : X \to Y$, $f_* : \pi_0(X) \to \pi_0(Y)$ is indeced by the universal property of coqualizers. For any two paths σ, τ with $\tau(0) = \sigma(1)$, we can define the composition of paths $\tau * \sigma$ just like the composition of homotopy in Fact 1.2. And we have known that such definition is up to homotopy in Remark 1.4. But here we give a different definition of $\tau * \sigma$ which will help us understand simplicial sets later.

Proposition 1.5. *$|\Lambda_k^n|$ is a strong deformation retract of $|\Delta^n|$.*

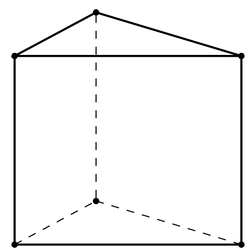
The proposition means there is a retraction $r : |\Delta^n| \to |\Lambda_k^n|$ such that $i \circ r \simeq id_{|\Delta^n|} \operatorname{rel} |\Lambda_k^n|$. The proof is intuitive. As $n = 2$ for example, the function r projects along the normal line of the missing interval. Then use the fact that $|\Delta^2|$ is convex to construct the homotopy of strong deformation retract.

There are some other important strong deformation retracts:

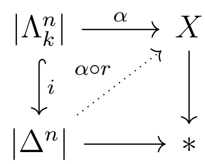
$$(|\Delta^n| \times \{0 \text{ (or } 1)\}) \cup (|\partial \Delta^n| \times I) \subseteq |\Delta^n| \times I$$

$$(|\Delta^n| \times \{0,1\}) \cup (|\Lambda^n_k| \times I) \subseteq |\Delta^n| \times I$$
$$(I^n \times \{0 \text{ (or } 1)\}) \cup (\partial I^n \times I) \subseteq I^n \times I$$

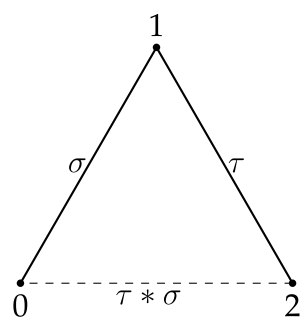
The proof is also intuitive, we also take $n = 2$ for example. Consider the following diagram



One of the faces is dug away in the picture. We fix a point on the top of the missing face and project along the line of the fixed point and a point contained in the interior of the triangular prism. And then use the convex property to construct the homotopy for strong deformation retracts. According to the property of retraction, we conclude that for all maps $\alpha : |\Lambda^n_k| \to X$, they can be extended to $|\Delta^n| \to X$.



Later, we will see every $X \to *$ is a *Serre fibration* and the lifting property in **Top** will be studied in detail there. To define the composition of paths, we take $n = 2$ and considedr $|\Lambda^2_1| \hookrightarrow |\Delta^2|$

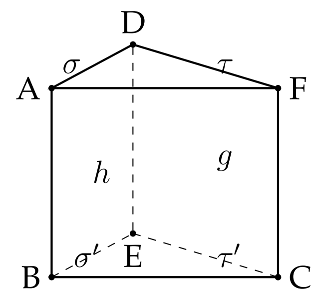


(2)

In the diagram above, $0 \to 1$ and $1 \to 2$ represent paths σ, τ respectively. Because of the lifting property of $|\Lambda^2_1| \hookrightarrow |\Delta^2|$, the two paths will induce a function on the whole $|\Delta^2|$. The new function restricting on $[0, 2]$ will represents a new path, which is denoted by $\tau * \sigma$. It's obvious to see, this definition of the composition of paths is the same as that we have talked above if we choose the standard retraction $r : |\Delta^2| \to |\Lambda^2_1|$ which is the projection along the middle line. But the lifting is actually arbitrary, and hence we need to prove it's unique up to homotopy. Note this composition above is defined for paths not for path classes. Therefore we also need to prove it's well defined for path classes. Note here the homotopy is the pointed homotopy.

Proposition 1.6. *The composition above between paths is defined up to homotopy, which means* $[\tau] \circ [\sigma] = [\tau * \sigma]$ *is well defined.*

Proof. If $h : \sigma \simeq \sigma' \text{ rel } |\partial\Delta^1|$, $g : \tau \simeq \tau' \text{ rel } |\partial\Delta^1|$ and σ, τ represent paths $x \to y, y \to z$ respectively, then we have the following diagram:



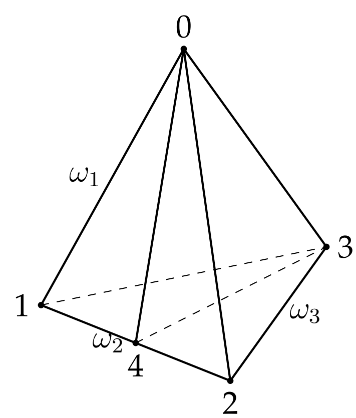
(3)

$[A, D], [D, F], [B, E], [E, C]$ represent paths $\sigma, \tau, \sigma', \tau'$ respectively. And via extensions on top and bottom faces, $[A, F], [B, C]$ represent paths $\tau \circ \sigma, \tau' \circ \sigma'$ respectively. Then use the fact that $(|\Delta^2| \times \{0, 1\}) \cup (|\Lambda^2_i| \times I) \subseteq |\Delta^2| \times I$ is a strong deformation retract to induce a function defined on the whole triangular prism. Especially its restriction on $[A, B, C, F]$ implies $\tau \circ \sigma \simeq \tau' \circ \sigma' \text{ rel } |\partial\Delta^1|$ □

Next, we want to prove the associativity of compositions.

Proposition 1.7. $x_0 \xrightarrow{\omega_1} x_1 \xrightarrow{\omega_2} x_2 \xrightarrow{\omega_3} x_3$ *is a chain consisting of three paths. Then* $\omega_3 * (\omega_2 * \omega_1) \simeq (\omega_3 * \omega_2) * \omega_1 \text{ rel } |\partial\Delta^1|$

Proof.



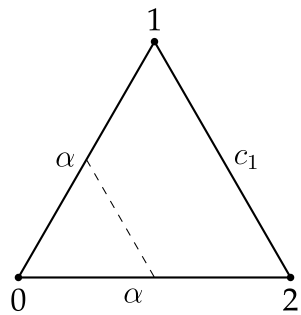
$P = 0 \to 1 \to 2 \to 3 \subseteq |\Delta^3|$

We first do extensions on $[0, 1, 2], [1, 2, 3]$ independently, which will induce $\omega_2 * \omega_1 : 0 \to 2$ and $\omega_3 * \omega_2 : 1 \to 3$. Projecting along the plane $[0, 4, 3]$, it's easy to see $[0, 1, 2] \cup [1, 2, 3] \subseteq [0, 1, 2, 3]$ is a strong deformation retract. Based on this retraction, the path on P can be naturally extended to the whole $[0, 1, 2, 3]$, whose restriction on $[0, 3]$ is denoted by $\omega_3 * \omega_2 * \omega_1$. This path actually factors through $0 \to 4 \to 3$. If we move the point 4 towards 2, this will give a homotopy $\omega_3 * \omega_2 * \omega_1 \simeq \omega_3 * (\omega_2 * \omega_1) \text{ rel } |\partial\Delta^1|$. On the other hand, if we move 4 towards 1, then it will imply $\omega_3 * \omega_2 * \omega_1 \simeq (\omega_3 * \omega_2) * \omega_1 \text{ rel } |\partial\Delta^1|$. □

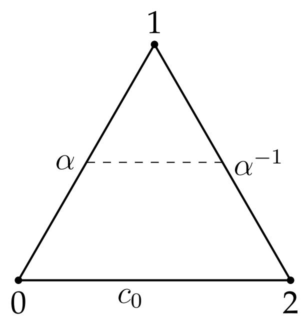
To prove the existence of the identity and inverses, we need to use Proposition 1.6.

Proposition 1.8. *The identity and inverses exist under the composition of path classes.*

Proof.



$0 \to 1$ and $0 \to 2$ represent the path α and $1 \to 2$ represents the constant path. Then the function can be extended from $|\partial\Delta^2|$ to $|\Delta^2|$ and it's constant on $\alpha(t)$ along the dotted line in the picture. Hence according to Proposition 1.6 $c_1 * \alpha \simeq \alpha \text{ rel } |\partial\Delta^1|$.



If we take a constant function on $\alpha(t) = \alpha(1-t)$ along the dotted line, then we will have $\alpha^{-1} * \alpha \simeq c_0 \text{ rel } |\partial\Delta^1|$. □

Now the *fundamental groupoid* $\Pi_1(X)$ of a space X is defined. Objects of $\Pi_1(X)$ are points of X and morphisms are path classes. The composition of morphisms is just the composition of path classes defined above. Then we see every morphism in $\Pi_1(X)$ is an isomorphism. We take the *fundamental group* $\pi_1(X, x)$ to be $\mathrm{Hom}_{\Pi_1(X)}(x, x)$. We can also view $\pi_1(X, x)$ as $[(|\Delta^1), |\partial\Delta^1|), (X, x)]$.

As a generalization of fundamental groups, we define higher homotopy groups as $\pi_n(X, x) = [(|\Delta^n|, |\partial\Delta^n|), (X, x)]$ where the homotopy is relative to $|\partial\Delta^n|$. For $|\Delta^n|/|\partial\Delta^n| \approx I^n/\partial I^n \approx S^n$, we have equations $\pi_n(X, x) = [(|\Delta^n|, |\partial\Delta^n|), (X, x)] = [(I^n, \partial I^n), (X, x)] = [(S^n, s), (X, x)]$.

To be convenient, we take $\pi_n(X, x) = [(I^n, \partial I^n), (X, x)]$. There are n's different group structures on $\pi_n(X, x)$. If $\alpha, \beta : (I^n, \partial I^n) \to (X, x)$ we can define the group structure as follows for $j = 1, ..., n$.

$$\beta *_j \alpha = \begin{cases} \alpha(..., 2t_j, ...), & t_j \le \frac{1}{2} \\ \beta(..., 2t_j - 1, ...), & \frac{1}{2} < t_j \le 1 \end{cases}$$

These group structures are well defined which is proved in the situation $n = 1$.

Theorem 1.9. *All the group structures on $\pi_n(X, x), n \geq 2$ are equal and commutative.*

Proof.

$$
\begin{aligned}
(a_1 *_i a_2) *_j (b_1 *_j b_2) &= (a_1 *_j b_1) *_i (a_2 *_j b_2) \\
&= \begin{cases} b_2(..., 2t_i, ..., 2t_j, ...), & t_i \leq \frac{1}{2}, t_j \leq \frac{1}{2} \\ b_1(..., 2t_i - 1, ..., 2t_j, ...), & \frac{1}{2} \leq t_i \leq 1, t_j \leq \frac{1}{2} \\ a_2(..., 2t_i, ..., 2t_j - 1, ...), & t_i \leq \frac{1}{2}, \frac{1}{2} \leq t_j \leq 1 \\ a_1(..., 2t_i - 1, ..., 2t_j - 1, ...), & \frac{1}{2} \leq t_i \leq 1, \frac{1}{2} \leq t_j \leq 1 \end{cases}
\end{aligned}
$$

Moreover these group structures have the same identity (the constant map). Therefore they are equal and commutative according to the following lemma. □

Lemma 1.10. *If on a group G, there are two group structures $\cdot$ and $*$ sharing the same identity element, and moreover if they satisfy $(u * v) \cdot (u' * v') = (u \cdot u') * (v \cdot v')$, then $\cdot = *$ and they are commutative.*

Proof. First, take $v = u' = 1$, $\Rightarrow u \cdot v' = u * v'$. Second, take $u = v' = 1$, $\Rightarrow v \cdot u' = u' * v = u' \cdot v$. □

We have known the definition of homotopy equivalence. Here we introduce a weaker concept *weak homotopy equivalence* which is more suitable to study the homotopy type of a space X.

Definition 1.11. *$f : X \to Y$ is a **weak equivalence** or a **weak homotopy equivalence** if for all $x \in X$, $f_* : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ and $f_* : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x)), \forall n \geq 1$.*

The homotopy equivalence is actually an equivalence, but the weak homotopy equivalence is not an equivalence relation, because in general given a map $f : X \to Y$ we may not construct a map $g : Y \to X$ to induce isomorphisms on all homotopy groups as inverses of f_*.

A famous theorem states every homotopy equivalence is a weak equivalence. To prove this we need more techniques.

If φ is a path in X from x_0 to x_1, we construct a map $\pi_n(X, x_0) \to \pi_n(X, x_1)$ as follows.

Let $\alpha : (I^n, \partial I^n) \to (X, x_0)$. We define $L' : (I^n \times \{0\}) \cup (\partial I^n \times I) \to X$ by $L'(x, t) = \begin{cases} \alpha(x), & t = 0 \\ \varphi(t), & x \in \partial I^n \end{cases}$. It's well defined. For $(I^n \times \{0\}) \cup (\partial I^n \times I) \subseteq I^n \times I$ is a strong deformation retract, there will be an extension $L : I^n \times I \to X$ and $L(-, 1)$ will define a map $(I^n, \partial I^n) \to (X, x_1)$. We denote this function by $T_\varphi(\alpha)$.

$$
\begin{array}{ccc}
I^n \times 0 \cup \partial I^n \times I & \xrightarrow{L' = (\alpha, \varphi \circ pr_2)} & X \\
\big\downarrow & \overset{\exists\, L}{\nearrow} & \\
I^n \times I & &
\end{array}
$$

Then $T_\varphi(\alpha) = L|I^n \times 1$. In the diagram above the dotted line can be $L' \circ r$ where $r : I^n \times I = I^{n+1} \to I^n \times 0 \cup \partial I^n \times I$ is the retraction.

Lemma 1.12. *The T defined above actually defines a functor from the fundamental groupoid of X to the category of groups, $\Pi_1(X) \to$ **Groups**. Hence, for any path class $[\varphi]$, $T_{[\varphi]}$ will define an isomorphism.*

Proof. At first, we prove T_φ is well defined. Given $h : \alpha \simeq \beta \text{ rel } \partial I^n$, we should prove $T_\varphi(\alpha) \simeq T_\varphi(\beta) \text{ rel } \partial I^n$ where $\alpha, \beta : (I^n, \partial I^n) \to (X, x_0)$, $\varphi : x_0 \to x_1$.

Since $(I^n \times \{0,1\} \times I) \cup (\partial I^n \times I \times I) \cup (I^n \times I \times \{0\}) = (\partial I^{n+1} \times I) \cup (I^{n+1} \times \{0\})$

$$\begin{array}{ccc} (I^n \times \{0,1\} \times I) \cup (\partial I^n \times I \times I) \cup (I^n \times I \times \{0\}) & \xrightarrow{\big((L_{\varphi,\alpha}, L_{\varphi,\beta}), \varphi\circ pr_3, h\big)} & X \\ \big\downarrow & \nearrow & \\ I^n \times I \times I & & \end{array}$$

Hence it can be extended to the whole $I^n \times I \times I$. Its restriction on $I^n \times I \times \{1\}$ will give the desire relative homotopy.

Now, we prove if $h : \varphi \simeq \phi \text{ rel } \partial I$, then $T_\varphi(\alpha) \simeq T_\phi(\alpha) \text{ rel } \partial I^n$.

$$\begin{array}{ccc} (I^n \times \{0,1\} \times I) \cup (\partial I^n \times I \times I) \cup (I^n \times I \times \{0\}) & \xrightarrow{\big((L_{\varphi,\alpha}, L_{\phi,\alpha}), h\circ pr_{I\times I}, \alpha\circ pr_1\big)} & X \\ \big\downarrow & \nearrow & \\ I^n \times I \times I & & \end{array}$$

The same process is available.

Hence T is well defined and the fact that it preserves identity and compositions can be proved using the same method above. □

Lemma 1.13. *If $f, g : X \to Y$, $H : X \times I \to Y$, $f \simeq g$, $\varphi = H|\{x\} \times I$ a path from $f(x)$ to $g(x)$, then we have the following commutative diagram:*

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{f_*} & \pi_n(Y, f(x)) \\ & \searrow^{g_*} & \big\downarrow T_\varphi \\ & & \pi_n(Y, g(x)) \end{array}$$

Proof. Given $\alpha : (I^n, \partial I^n) \to (X, x)$.

$G : I^n \times I \xrightarrow{\alpha\times \mathrm{id_I}} X \times I \xrightarrow{H} Y$ gives a homotopy $f \circ \alpha \simeq g \circ \alpha$. On $\partial I^n \times I$, $G(x', t) = H \circ (\alpha \times \mathrm{id})(x', t) = H(x, t) = \varphi(t)$. We should prove $T_\varphi(f \circ \alpha) \simeq g \circ \alpha, \ \mathrm{rel} \partial I^n$.

$$\begin{array}{ccc} (I^n \times \{0,1\} \times I) \cup (\partial I^n \times I \times I) \cup (I^n \times I \times \{0\}) & \xrightarrow{\big((L_{\varphi,f\alpha}, G), \varphi pr_3, f\alpha pr_1\big)} & Y \\ \big\downarrow & \nearrow & \\ I^n \times I \times I & & \end{array}$$

The restriction of the extension to $I^n \times I \times 1$ gives a homotopy $T_\varphi(f \circ g) \simeq g \circ \alpha \operatorname{rel} \partial I^n$. □

Theorem 1.14. *Every homotopy equivalence is a weak equivalence.*

Proof. Given $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y, g \circ f \simeq \mathrm{id}_X$, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_n(X,x) & \xrightarrow{g_* f_*} & \pi_n(X, g \circ f(x)) \\ & \searrow^{\mathrm{id}} & \downarrow T \\ & & \pi_n(X,x) \end{array}$$

Hence f_* is monic. The similar diagram will imply f_* is epic. Hence it's an isomorphism. □

1.3 Adjoint Functors in Top

Definition 1.15. *In* **Top**, $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ *consists of continuous maps between X and Y. In it we can define the* ***compact-open topology*** *as follows. Let $K \subseteq X$ be compact and $U \subseteq Y$ be open.*

$$W(K,U) = \{f : X \to Y \in \mathrm{Hom}_{\mathbf{Top}}(X,Y) | f(K) \subseteq U\}$$

$W(K,U)$'s form a subbasis of this topology which means $\bigcap_{\text{finite}} W(K,U)$ forms a basis. With this topology the mapping space is denoted by Y^X.

Remark 1.16. Given a continuous map $f : X \to Y$, it induces the function $f^* : Z^Y \to Z^X$ by compositions. We prove it's continuous.

Proof. Assume $W(K,U) \subseteq Z^X$ where K is a compact subset in X and U is an open subset in Z.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$g \in f^{*-1}(W(K,U))$ iff $f^*(g) = g \circ f \in W(K,U)$ iff $g \circ f(K) \subseteq U$ iff $g \in W(f(K),U)$. □

Remark 1.17. Dually, f_* is continuous as well. Consider $f_* : X^Z \to Y^Z$ and $W(K,U) \subseteq Y^Z$.

$$Z \xrightarrow{g} X \xrightarrow{f} Y$$

$g \in f_*^{-1}(W(K,U))$ iff $f_*(g) = f \circ g \in W(K,U)$ iff $f \circ g(K) \subseteq U$ iff $g(K) \subseteq f^{-1}(U)$ iff $g \in W(K, f^{-1}(U))$.

Remark 1.18. In **Sets**, $Y^X = \mathrm{Hom}_{\mathbf{Sets}}(X,Y)$, then we will have the following adjunction

$$\mathrm{Hom}_{\mathbf{Sets}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, Z^Y)$$

where is isomorphism is defined by the evaluation. In categorical notions, this means **Sets** is a *closed symmetric monoidal category* and **Cat** belongs to this class as well. For any closed symmetric monoidal category $\mathcal{V}$ it's enriched on itself. In $\mathcal{V}$, the tensor product

$- \otimes X$ admits a right adjoint $\mathrm{Map}(X, -)$. Note in the case of **Sets**, $- \otimes X$ is just $- \times X$ and $\mathrm{Map}(X, -)$ is just $(-)^X$. The enriched composition morphism

$$\circ : \mathrm{Map}(Y, Z) \otimes \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$$

is defined to be the dual of compositions of

$$\mathrm{Map}(Y, Z) \otimes \big(\mathrm{Map}(X, Y) \otimes X\big) \xrightarrow{\mathrm{id} \otimes\, ev} \mathrm{Map}(Y, Z) \otimes Y \xrightarrow{ev} Z$$

where ev is dual to the identity map of $\mathrm{Map}(X, Y)$ via the adjunction.

In general **Top** doesn't have the property stated in Remark 1.18 which means some $- \times X$ does not admit the right adjoint functor but for some special topological spaces X it's true.

Definition 1.19. *A Hausdorff space X is **locally compact** if for any point $x \in X$ and any open neighborhood U containing it, there will exist an open neighborhood of x such that $V \subseteq \overline{V} \subseteq U$ and the closure $\overline{V}$ is compact.*

Remark 1.20. There is a more general notion of locally compactness for spaces not necessarily Hausdorff. But under the assumption of Hausdorff property it's equivalent to that we state here. Why we choose this definition? In general talking about locally compact space we are often interested in Hausdorff space such as $\mathbb{R}^n$ and manifolds

Proposition 1.21. *Suppose X is locally compact Hausdorff. Then the evaluation map*

$$ev = e_{X,Y} : Y^X \times X \to Y,\ (f, x) \mapsto f(x)$$

is continuous.

Proof. Let $U \subseteq Y$ be an open subset. If $(f, x) \in ev^{-1}(U)$, then $f(x) \in U$ and $x \in f^{-1}(U)$. Since X is locally compact Hausdorff, there will be an open nerighborhood V of x such that $V \subseteq \overline{V} \subseteq f^{-1}(U)$ where $\overline{V}$ is compact. Hence $f(\overline{V}) \subseteq U$ which means $f \in W(\overline{V}, U)$. Then $(f, x) \in W(\overline{V}, U) \times V \subseteq ev^{-1}(U)$. □

Theorem 1.22. *For any locally compact Hausdorff space Y there is an adjuncion*

$$- \times Y : \mathbf{Top} \rightleftarrows \mathbf{Top} : (-)^Y$$

and natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}}(X, Z^Y)$$

Proof. The adjuncion is induced from that in **Sets** and it's enough to prove $f : X \times Y \to Z$ is continuous iff $\tilde{f} : X \to Z^Y$ is continuous as well.

"$\Rightarrow$". Suppose f is continuous and $W(K, U) \subseteq Z^Y$. $x \in \tilde{f}^{-1}(W(K, U))$ iff $\tilde{f}(x) \in W(K, U)$ iff $f(x, y) \in U$ for all $y \in K$ iff $\{x\} \times K \subseteq f^{-1}(U)$. For any $y \in K$ there will be open subsets V_y, V'_y such that $(x, y) \in V_y \times V'_y \subseteq f^{-1}(U)$. Since K is compact, there are only finitely many V_y''s covering K. Let $V' = \bigcup_{\text{finite}} V_y'$ and $V = \bigcap_{\text{finite}} V_y$. $V \times V_y' \subseteq V_y \times V_y' \subseteq f^{-1}(U)$. Then $V \times V' = \cup(V \times V_y') \subseteq f^{-1}(U)$ while $K \subseteq V'$. We see $x \in V \subseteq \tilde{f}^{-1}(W(K,U))$.

"$\Leftarrow$". Assume $\tilde{f}$ is continuous.

$$f : X \times Y \xrightarrow{\tilde{f}\times \mathrm{id}_Y} Z^Y \times Y \xrightarrow{e_{Y,Z}} Z$$

From Proposition 1.21 we know it's continuous. $\square$

Example 1.23. The unit interval $I = [0,1]$ is locally compact Hausdorff. Hence there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(X \times I, Y) \cong \mathrm{Hom}_{\mathbf{Top}}(X, Y^I)$$

Similar to the diagram in Definition 1.1 here we can defined the concept of *cohomotopy* or *right homotopy* (this name comes from homotopical algebra, see Definition 2.33) using the following diagram

$$\begin{array}{ccccc} & & Y^I & & \\ & \overset{\widetilde{H}}{\nearrow} & \downarrow{\scriptstyle (e_0,e_1)} & \overset{s}{\underset{\sim}{\nwarrow}} & \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xleftarrow[\Delta]{} & Y \end{array}$$

$f \simeq_r g$ if such $\widetilde{H}$ exists. Note

$$\begin{cases} \widetilde{H} : X \to Y^I \\ e_0 \circ \widetilde{H} = f \\ e_1 \circ \widetilde{H} = g \end{cases}$$

actually means $H : X \times I \to Y$, $H|X \times 0 = f$ and $H|X \times 1 = g$ which is just the definition of homotopy.

In the diagram above s is the constant map sending y to the constant function $y : I \to Y$. In the following we will prove s is a weak homotopy equivalence and (e_0, e_1) is a *Serre fibration* (see Definition 1.72) (in fact it's a *Hurewicz fibration*, see Example 1.62 (1)). Then it's actually the diagram of right homotopy in the model category of **Top**.

Now we start to prove s is a weak homotopy equivalence. From Example 1.62 (2), we know e_0 is a Hurewicz fibration. Here we want show it's a trivial Serre fibration, so that after proving this since $\mathrm{id}_Y = e_0 \circ s$, we conclude s is a weak homotopy equivalence.

According to Proposition 1.77, it's enough to show e_0 has the right lifting property with respect to all $|\partial\Delta^n| \hookrightarrow |\Delta^n|$. But from the adjoint pair $- \times I$ and $(-)^I$, the following two lifting problems are equivalent

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & Y^I \\ \big\downarrow & \nearrow & \big\downarrow {\scriptstyle e_0} \\ |\Delta^n| & \longrightarrow & Y \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccc} |\partial\Delta^n| & & \longrightarrow & & |\Delta^n| \\ {\scriptstyle i_0}\big\downarrow & & & & \big\downarrow \\ |\partial\Delta^n| \times I & & \longrightarrow & & (|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times I) \\ & & & & \big\downarrow \quad \searrow \\ & & & & |\Delta^n| \times I \xrightarrow{\exists!} Y \end{array}$$

The second one is trivially solved since $(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times I) \hookrightarrow |\Delta^n| \times I$ is a strong deformation retract.

Theorem 1.24. *If X and Y are both locally compact Hausdorff, then the adjunction described in Theorem 1.22 is a homeomorphism.*

$$Z^{X\times Y} \approx (Z^Y)^X$$

Note we can in fact only assume X is Hausdorff and such proof can be found in [Hat02] Proposition A.16.

Proof. First we prove $\widetilde{(-)} : Z^{X\times Y} \to (Z^Y)^X,\ f \mapsto \tilde{f}$ is continuous. Since X and Y are locally compact Hausdorff, $X \times Y$ is locally compact Hausdorff as well. This can be checked directly. Then the map

$$ev = e_{X\times Y,Z} : Z^{X\times Y} \times (X \times Y) \to Z$$

is continuous. Then

$$\tilde{ev} : Z^{X\times Y} \times X \to Z^Y$$

is continuous as well from Theorem 1.22. Still from Theorem 1.22

$$\tilde{\tilde{ev}} : Z^{X\times Y} \to (Z^Y)^X$$

is continuous. It's obviously to see $\tilde{\tilde{ev}} = \widetilde{(-)}$.

Next we prove it's a homeomorphism.

$$\mathrm{Hom}_{\mathbf{Top}}(A, Z^{X\times Y}) \cong \mathrm{Hom}_{\mathbf{Top}}(A \times X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}}(A \times X, Z^Y) \cong \mathrm{Hom}_{\mathbf{Top}}(A, (Z^Y)^X)$$

The bijection is induced by the continuous map $\widetilde{(-)}$. Using Yoneda's lemma, we see it's actually an isomorphism in **Top** hence a homeomorphism. □

Remark 1.25. Note that the step to prove $\widetilde{(-)}$ is continuous can't be omitted, since it means this function is actually in **Top**. We can use Yoneda's lemma, only if this is true.

Corollary 1.26. *If Y is locally compact Hausdorff, then there is an adjunction at the level of homotopy categories.*

$$- \times Y : \mathbf{HoTop} \rightleftarrows \mathbf{HoTop} : (-)^Y$$

Proof. At first we should prove two functors are well defined, which means $- \times Y$ and $(-)^Y$ preserve homotopies. If $H : X \times I \to X',\ f \simeq g$, then $H \times \mathrm{id}_Y : X \times I \times Y \to X' \times Y,\ f \times \mathrm{id}_Y \simeq g \times \mathrm{id}_Y$. For $(-)^Y$, consider

$$X^Y \times I \xrightarrow{\mu} (X \times I)^Y \xrightarrow{H_*} X'^Y$$

where the first function is $(\varphi, t) \mapsto (y \mapsto (\varphi(y), t))$ and the second function is defined by composition. The second one is continuous according to Remark 1.17. Hence we only need to prove μ is continuous as well. Assume K is compact in Y and U is open in $X \times I$. $(\varphi, t) \in \mu^{-1}(W(K, U))$ iff $(y \mapsto (\varphi(y), t)) \in W(K, U)$ iff $(\varphi(K), t) \subseteq U$. Note here t is fixed.

For any $x \in \varphi(K)$ there will be open subsets V_x and V'_x such that $(x,t) \in V_x \times V'_x \subseteq U$. Since $\varphi(K)$ is compact, we can choose finitely many of them and let $V = \bigcup_{\text{finite}} V_x \supseteq \varphi(K)$, $V' = \bigcap_{\text{finite}} V'_x$. Then $V \times V' \subseteq U$ and $(\varphi, t) \in W(K,V) \times V' \subseteq \mu^{-1}(W(K,U))$.

Next we need to prove the adjunction in Theorem 1.22 passes to homotopy. Suppose $f, g : X \times Y \to Z$ such that there is a homotopy $H : X \times Y \times I \to Z,\ f \simeq g$. Then $\widetilde{H} : X \times I \to Z^Y,\ \tilde{f} \simeq \tilde{g}$. The converse is true as well. □

Now let us talk about some universal constructions in $\mathbf{Top}_*$. There coproducts are *wedge products*, $X \vee Y = \frac{X \coprod Y}{x_0 \sim y_0}$ and products are defined as usual. But apart from usual products here is a special concept *smash product* which is defined to be

$$X \wedge Y = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y} = \frac{X \times Y}{X \vee Y}$$

where $X \vee Y \hookrightarrow X \times Y$ is $x \mapsto (x, y_0),\ y \mapsto (x_0, y)$.

Example 1.27. Identify S^1 with $I/\partial I$, $S^1 \wedge S^1 = \frac{I \times I}{I \times \{0,1\} \cup \{0,1\} \times I} = I^2/\partial I^2 \approx S^2$.

Remark 1.28. Why we introduce the concept of smash products here? At the level of $\mathbf{Sets}$, usual products $- \times Y$ doesn't admit a right adjoint $(-)^Y$. For Z^Y, the base point is the canstant map $z_0 : Y \to Z$. Hence in $\mathrm{Hom}_{\mathbf{Sets}_*}(X, Z^Y)$ we should require for $x_0 \in X$ the induced morphism $Y \to Z$ is constant and for any $x \in X$ morphisms $Y \to Z$ should send y_0 to z_0. This argument proves there is an ajunction in $\mathbf{Sets}_*$.

$$- \wedge Y : \mathbf{Sets}_* \rightleftarrows \mathbf{Sets}_* : (-)^Y$$

And it's true as well in $\mathbf{Top}_*$ if Y is locally compact Hausdorff. Moreover this adjuncion can also pass to homotopy categories.

Remark 1.29. You should be careful. In general smash products are not associative which is different from the case of usual products. But from the definition we see it's commutative, and if $X,\ Z$ are locally compact Hausdorff, then associative law is valid and we have homeomorphisms

$$(X \wedge Y) \wedge Z \approx X \wedge (Y \wedge Z) \approx \frac{X \times Y \times Z}{\{x_0\} \times Y \times Z \cup X \times \{y_0\} \times Z \cup X \times Y \times \{z_0\}}$$

This can be proved using Yoneda's lemma just like Theorem 1.24.

Remark 1.30. For any pointed space X, the *cylinder* is defined to be

$$\mathrm{Cyl}(X) := \frac{X \times I}{\{x_0\} \times I}$$

This construction characterize pointed homotopy, since any pointed homotopy $H : X \times I \to Y$ factors through $\mathrm{Cyl}(X)$.

The *cone* is defined to be

$$\mathrm{Cone}(X) := \frac{X \times I}{\{x_0\} \times I \cup X \times 0}$$

and any pointed homotopy starting from a constant maps factors through $\mathrm{Cone}(X)$. The final construction is called *suspension*

$$\Sigma X := X \wedge S^1 = \frac{X \times I}{\{x_0\} \times I \cup X \times \partial I}$$

All three constructions above are functorial in $\mathbf{Top}_*$ and admit right adjoints. For simplicity you can find them in $\mathbf{Sets}_*$ and they are all valid in $\mathbf{Top}_*$ even in $\mathbf{HoTop}$ since I is locally compact Hausdorff. In Remark 1.28 we have described the adjunction induced by $- \wedge Y$ and $(-)^Y$. Here $Y = S^1$ and the functor $- \wedge S^1$ is denoted by Σ. We write the functor $(-)^{S^1}$ as Ω, which is called the *loop space* functor. Then we have the adjunction

$$\Sigma : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : \Omega$$

For $\mathrm{Cyl}(X)$, its right adjoint functor is $\mathrm{Hom}_{\mathbf{Sets}}(I, -)$ and for any set $\mathrm{Hom}_{\mathbf{Sets}}(I, Z)$ its base point is the constant map. Note it's different from that in Remark 1.28 and there the right adjoint functor is actually $\mathrm{Hom}_{\mathbf{Sets}_*}(I, -)$. The functor $\mathrm{Hom}_{\mathbf{Sets}}(I, -)$ is also denoted by $(-)^I$ and this will not make confusion since we rarely talk about it in the following. Then we have the adjunction

$$\mathrm{Cyl} : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : (-)^I$$

As for $\mathrm{Cone}(X)$, we see functions $\tilde{f} : X \to Y^I$ from $f : \frac{X \times I}{\{x_0\} \times I \cup X \times 0} \to Y$ should satisfy sending x_0 to constant map and for any x, $\tilde{f}(x)$ sending 0 to the base point $y_0 \in Y$. Hence its right adjoint functor is defined to be the sub-path spaces with the initial point fixed $\mathrm{Path}_0(X) := \{\varphi : I \to X | \varphi(0) = x_0\}$.

The suspension functor and loop space functor are important in algebriac topology and they are the starting point of *stable homotopy theory*. Later we will see there is a generalization of them in *pointed model categories*.

$S^0 = \partial I = \{0, 1\}$. Then $S^0 \wedge X = \frac{X \times \partial I}{\{x_0\} \times \partial I \cup X \times 0}$. We can define the embedding $X \hookrightarrow S^0 \wedge X,\ x \mapsto (x, 1)$ which is actually a homeomorphism. Note $X \times \partial I = X \coprod X$. From this point the homeomorphism will be clear. Hence $S^0 \wedge X \approx X$, especially $S^0 \wedge S^1 \approx S^1$.

Theorem 1.31. *$\Sigma^n X \approx X \wedge S^n \approx X \wedge I^n / \partial I^n$ for $n \geq 0$.*

Proof. $X \wedge S^n \approx X \wedge I^n / \partial I^n$ is clear since $S^n \approx I^n / \partial I^n$. We assume $\Sigma^n X \approx I^n / \partial I^n$ holds and then

$$\begin{aligned} \Sigma^{n+1} X = \Sigma(\Sigma^n X) &= \Sigma(X \wedge I^n / \partial I^n) \\ &= (X \wedge I^n / \partial I^n) \wedge I / \partial I \\ &\approx X \wedge (I^n / \partial I^n \wedge I / \partial I) \end{aligned}$$

since S^n and S^1 are locally compact Hausdorff and from Remark 1.29 we see in this case smash products are associative.

$$I^n/\partial I^n \wedge I/\partial I = \frac{I^n \times I}{\partial I^n \times I \cup I^n \times \partial I} = I^{n+1}/\partial I^{n+1}$$

We conclude $\Sigma^{n+1}X \approx X \wedge I^{n+1}/\partial I^{n+1}$. □

Corollary 1.32. *$\Sigma^n S^0 \approx S^0 \wedge S^n \approx S^n$ and $\Sigma S^n \approx S^{n+1}$.*

The collection of spheres $\{S^n\}$ forms the *sphere spectrum*.

Definition 1.33. *A **spectrum** X is a collection of pointed spaces X_0 for $n \geq 0$ with structure maps*

$$\sigma_n^X : \Sigma X \to X_{n+1}$$

or with dual maps $\tilde{\sigma}_n^X : X_n \to \Omega X_{n+1}$. A Ω-spectrum is a spectrum such that all $\tilde{\sigma}_n^X$ are weak homotopy equivalences.

We will not study spectrums in detail here and we advise readers to read [BaR20] if they are interested in stable homotopy theory.

Remark 1.34. In $\mathbf{Top}_*$ the pointed homotopy is just the homotopy relative to the base point. We know $\pi_n(X, x_0)$ is defined to be the relative homotopy class of maps between S^n and X. Hence passing to $\mathbf{HoTop}_*$

$$\pi_n(X, x_0) = [S^n, X]_* = [\Sigma^n S^0, X]_* = [S^0, \Omega^n X]_* = \pi_0(\Omega^n X)$$

In general, we can define a group structure on $\mathrm{Hom}_{\mathbf{HoTop}}(\Sigma X, Y) = [\Sigma X, Y]_*$. $\Sigma X = \frac{X \times I}{\{x_0\} \times I \cup X \times \partial I}$. For any two $f, g : \Sigma X \to Y$ we let

$$(g * f)(x, t) = \begin{cases} f(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ g(x, 2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and the map $\Sigma : [\Sigma X, Y]_* \to [\Sigma^2 X, \Sigma Y]_*$ is a group homomorphism. Using Lemma 1.10, we see for $n \geq 2$, $[\Sigma^n X, Y]_*$ is abelian where $\Sigma^n X = X \wedge I^n/\partial I^n = \frac{X \times I^n}{\{x_0\} \times I^n \cup X \times \partial I^n}$. There is a natural question when the map Σ is a group isomorphism. It's clear when computing homotopy groups on spheres.

Using the *homotopy excision theorem* we can prove the *suspension theorem* which states $\Sigma : \pi_i(S^n) = [S^i, S^n]_* \to [\Sigma S^i, \Sigma S^n]_* = \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$, an epimorphism for $i \geq 2n-1$. Proofs can be found in [tom08] Section 6.4.

1.4 Fiber and Cofiber Sequences

In $\mathbf{Sets}_*$ the sequence

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

is called exact if $f(X) = g^{-1}(z_0)$.

Example 1.35. In $\mathbf{Sets}_*$, $0 \to (X, x_0) \xrightarrow{f} (Y, y_0)$ don't mean f is injective. For example let $(X, x_0) = (\{0,1,2\}, 0)$ and $(Y, y_0) = (\{0,1\}, 0)$, $f(1) = f(2) = 1$. Then this sequence will be exact but f is not injective. This is different from that in an abelian category.

Definition 1.36. *A sequence of pointed spaces $A \xrightarrow{f} B \xrightarrow{g} C$ is **h-coexact** if for any pointed space Z, the following sequence*

$$[C, Z]_* \xrightarrow{g^*} [B, Z]_* \xrightarrow{f^*} [A, Z]_*$$

*is exact. Dually it's called **h-exact** if the sequence*

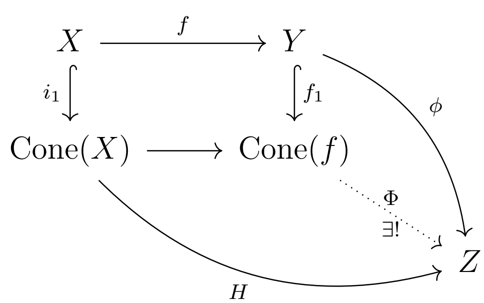
$$[Z, A]_* \xrightarrow{f_*} [Z, B]_* \xrightarrow{g_*} [Z, C]_*$$

is exact. Here "h" means "pointed homotopy".

Fact 1.37. We give more concrete explanations of h-coexactness here.

Given a pointed map $\phi : B \to Z$, $f^*(\phi) = \phi \circ f : A \to Z$ is null homotopic which means $\phi \circ f \simeq$ constant map iff $\exists \varphi : C \to Z$ such that $\varphi \circ g \simeq \phi$. Let $Z = C$, $\varphi = \mathrm{id}_C$, $\phi = g$. Then this means $g \circ f \simeq$ constant map, which is similar to the case of complexes. Note that here all homotopies are pointed.

Definition 1.38. *For a map $f : X \to Y$ the **mapping cone** is defined as the following pushout*



Remark 1.39. Recall in Remark 1.30, $\mathrm{Cone}(X) = \frac{X \times I}{\{x_0\} \times I \cup X \times 0}$. Here $i_1(x) = (x, 1)$. Then

$$\mathrm{Cone}(f) = \frac{\mathrm{Cone}(X) \coprod Y}{(x,1) \sim f(x)}$$

and $f_1(y) = y$. Consider the sequence

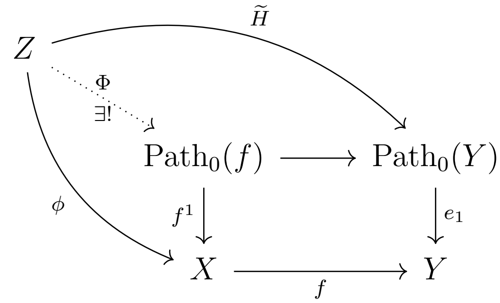
$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f)$$

Given a map $\phi : Y \to Z$ such that $f^*(\phi) = \phi \circ f : X \to Z$ is null homotopic. Let $H : X \times I \to Z$, constant map $\phi \circ f$. H factors through $\mathrm{Cone}(X)$. Then in the pushout diagram there will be a unique map $\Phi : \mathrm{Cone}(f) \to Z$ satisfying $\Phi \circ f_1 = \phi$. Hence

$$[\mathrm{Cone}(f), Z]_* \xrightarrow{f_1^*} [Y, Z]_* \xrightarrow{f^*} [X, Z]_*$$

is exact and the original sequence is h-coexact.

Definition 1.40. *For a map $f : X \to Y$, the* ***mapping path space*** *is defined as the following pullback*



Remark 1.41. Here

$$\begin{aligned}\mathrm{Path}_0(f) &= X \times_Y \mathrm{Path}_0(Y)\\ &= \{(x,\varphi)|f(x) = \varphi(1),\ \text{where } \varphi : I \to Y \text{ with } \varphi(0) = y_0\}\end{aligned}$$

From Example 1.23 we know the homotopy and cohomotopy are equivalent. Consider the sequence

$$[Z, \mathrm{Path}_0(f)]_* \xrightarrow{f^1_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Given a map $\phi : Z \to X$ and a pointed homotopy $H : Z \times I \to Y,$ constant map $\simeq f \circ \phi$. Then H indeuces $\widetilde{H} : Z \to \mathrm{Path}_0(Y)$ with $e_1 \circ \widetilde{H}(z) = H(z, 1) = f \circ \phi(z)$. Therefore there exists the unique $\Phi : Z \to \mathrm{Path}_0(f)$ satisfying $f^1 \circ \Phi = \phi$. Then the sequence above is exact and

$$\mathrm{Path}_0(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

is h-exact.

In general there are following long h-coexact and h-exact sequences.

Theorem 1.42 (Cofiber Sequence or Pupp Sequence)**.** *For any pointed map $f : (X, x_0) \to (Y, y_0)$, the h-coexact sequence $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f)$ induces a long h-coexact sequence of pointed spaces*

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{\partial} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma\mathrm{Cone}(f) \xrightarrow{\Sigma\partial} \ldots \tag{4}$$

where

$$\partial = p(f) : \mathrm{Cone}(f) \twoheadrightarrow \Sigma X = \frac{X \times I}{\{x_0\} \times I \cup X \times \partial I} = \mathrm{Cone}(X)/i_1(X) = \mathrm{Cone}(f)/f_1(Y)$$

is the canonical quotient map. Then for any pointed space (Z, z_0), there is a long exact sequence

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{f_1^*} [\mathrm{Cone}(f), Z]_* \xleftarrow{\partial^*} [\Sigma X, Z]_* \xleftarrow{\Sigma f^*} [\Sigma Y, Z]_* \xleftarrow{\Sigma f_1^*} \ldots$$

Dually there is a theorem for the fiber sequence.

Theorem 1.43 (Fiber Sequence). *For any pointed map $f : (X, x_0) \to (Y, y_0)$, the h-exact sequence* $\mathrm{Path}_0(f) \xrightarrow{f^1} X \xrightarrow{f} Y$ *induces a long h-exact sequence of pointed spaces*

$$... \xrightarrow{\Omega i(f)} \Omega\mathrm{Path}_0(f) \xrightarrow{\Omega f_1} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i(f)} \mathrm{Path}_0(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

where $i(f)$ sends a loop φ to (x_0, φ).

We only prove Theorem 1.42 and Theorem 1.43 is left to readers since all things are dual.

Proof of Theorem 1.42. We use the long h-coexact sequence we have proved in Remark 1.39 to prove this theorem.

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} ...$$

We have the following commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\quad f \quad} & Y & \xhookrightarrow{\quad i_1 \quad} & \mathrm{Cone}(Y) \\
{\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle j_1} \\
\mathrm{Cone}(X) & \xrightarrow{\quad j \quad} & \mathrm{Cone}(f) & \xhookrightarrow{\quad f_2 \quad} & \mathrm{Cone}(f_1) \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p(f)} & & {\scriptstyle r(f)}\big\uparrow\big\downarrow{\scriptstyle q(f)} \\
\Sigma X \approx \mathrm{Cone}(X)/\mathrm{im}\ i_1 & \longrightarrow & \Sigma X \approx \mathrm{Cone}(f)/\mathrm{im}\ f_1 & \longrightarrow & \Sigma X \approx \mathrm{Cone}(f_1)/\mathrm{im}\ j_1
\end{array}$$

where

$$\mathrm{Cone}(f_1) = \frac{\mathrm{Cone}(Y) \coprod \mathrm{Cone}(f)}{(y,1) \sim f_1(y) = y} = \frac{\mathrm{Cone}(Y) \coprod \mathrm{Cone}(X) \coprod Y}{(y,1) \sim y,\ (x,1) \sim f(x)} = \frac{\mathrm{Cone}(Y) \coprod \mathrm{Cone}(X)}{(x,1) \sim (f(x),1)}$$

then $\mathrm{Cone}(f_1)/\mathrm{Cone}(Y) = \mathrm{Cone}(X)/X \times 1 = \Sigma X$.

We first show $q(f)$ is a pointed homotopy equivalence. Define

$$r(f) : \Sigma X \to \mathrm{Cone}(f_1),\ (x,s) \mapsto \begin{cases} (x, 2s), & 0 \le s \le \frac{1}{2} \\ (f(x), 2-2s), & \frac{1}{2} \le s \le 1 \end{cases}$$

We see $r(f) \circ q(f) : \mathrm{Cone}(f_1) \to \mathrm{Cone}(f_1)$ is that

$$(y,s) \mapsto (f(x), 0) = (y, 0) = *,\quad (x,s) \mapsto \begin{cases} (x, 2s), & 0 \le s \le \frac{1}{2} \\ (f(x), 2-2s), & \frac{1}{2} \le s \le 1 \end{cases}$$

The pointed homotopy $H : \mathrm{Cone}(f_1) \times I \to \mathrm{Cone}(f_1),\ \mathrm{id}_{\mathrm{Cone}(f_1)} \simeq r(f) \circ q(f)$ is defined as

$$(y,s,t) \mapsto (y, (1-t)s),\ (x,s,t) \mapsto \begin{cases} (x, (1+t)s), & 0 \le s \le \frac{1}{1+t} \\ (f(x), 2-(1+t)s), & \frac{1}{1+t} \le s \le 1 \end{cases}$$

On the other hand $q(f) \circ r(f) : \Sigma X \to \Sigma X$ is just $(x,s) \mapsto \begin{cases} (x, 2s), & 0 \le s \le \frac{1}{2} \\ (s, 1), & \frac{1}{2} \le s \le 1 \end{cases}$. Then we can define the pointed homotopy $K : \Sigma X \times I \to \Sigma X,\ \mathrm{id}_{\Sigma X} \simeq q(f) \circ r(f)$ as follows

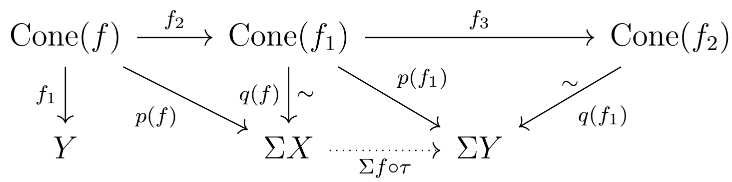
$$(x,s,t) \mapsto \begin{cases} (x, (1+t)s), & 0 \le s \le \frac{1}{1+t} \\ *, & \frac{1}{1+t} \le s \le 1 \end{cases}$$

These prove $q(f)$ is pointed homotopy equivalence.

Consider

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \ldots$$

This induces the following diagram where $\tau : \Sigma x \to \Sigma X,\ (x,s) \mapsto (x, 1-s)$ is a homeomorphism such that $\tau \circ \tau = \mathrm{id}$ and $\tau \circ \Sigma f = \Sigma f \circ \tau$.



Next we prove this diagram is homotopy commutative which means $\Sigma f \circ \tau \circ q(f) \simeq p(f_1)$ $\Leftrightarrow \Sigma f \circ \tau \simeq p(f_1) \circ r(f) \Leftrightarrow \Sigma f \simeq p(f_1) \circ r(f) \circ \tau$.

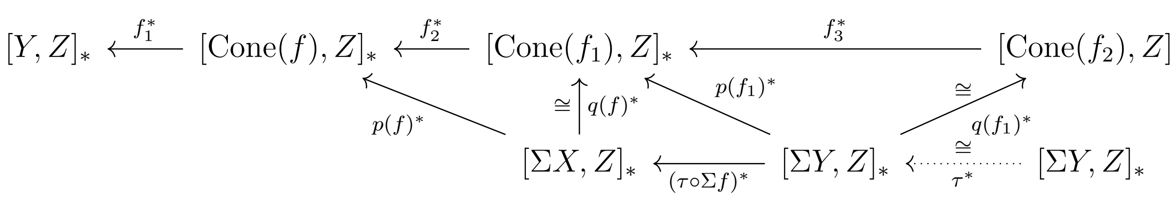
Note that $p(f_1) \circ r(f) \circ \tau : \Sigma X \to \Sigma Y$ is actually

$$(x,s) \mapsto (x, 1-s) \mapsto \begin{cases} (f(x), 2s), & 0 \le s \le \frac{1}{2} \\ (x, 2-2s), & \frac{1}{2} \le s \le 1 \end{cases} \mapsto \begin{cases} (f(x), 2s), & 0 \le s \le \frac{1}{2} \\ *, & \frac{1}{2} \le s \le 1 \end{cases}$$

and we can define the pointed homotopy to be

$$L : \Sigma X \times I \to \Sigma Y,\ (x,s,t) \mapsto \begin{cases} \big(f(x), (1+t)s\big), & 0 \le s \le \frac{1}{1+t} \\ *, & \frac{1}{1+t} \le s \le 1 \end{cases}$$

Hence this diagram is commutative in $\mathbf{HoTop}_*$. Then we apply the functor $[-, Z]_*$ to this diagram. We obtain



Note that the top sequence is exact since $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \ldots$ is h-coexact. Then the following sequence is exact

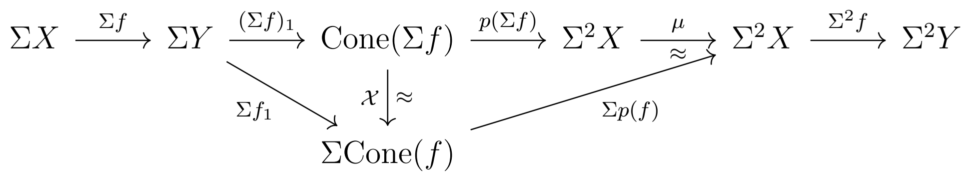
$$[Y, Z]_* \xleftarrow{f_1^*} [\mathrm{Cone}(f), Z]_* \xleftarrow{p(f)^*} [\Sigma X, Z]_* \xleftarrow{\Sigma f^*} [\Sigma Y, Z]_*$$

then

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{(\Sigma f)_1} \mathrm{Cone}(\Sigma f)$$

is h-coexact.

Be careful here! In the diagram $\mathrm{Cone}(\Sigma f)$ is different from $\Sigma\mathrm{Cone}(f)$. To obtain the sequence in the Theorem 1.43 we need to prove they are homeomorphic. Actaully we prove we have the following commutative diagram

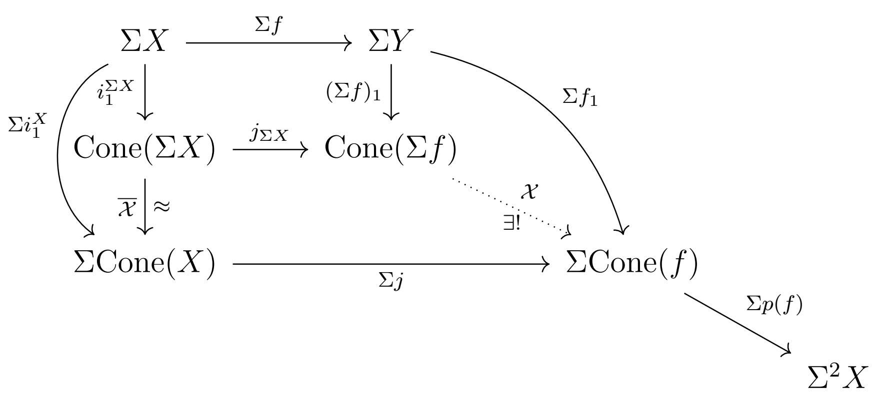


where $\mu : \Sigma^2 X \to \Sigma^2 X,\ (x, s, t) \mapsto (x, t, s)$. If we prove this, then we succeed in extending the h-coexact sequence to the level of $n = 2$. If we have extended the sequence to the level of n, you can apply the functor Σ^{n-1} to the diagram above or you can understand it that we replace f by Σf. Finally we extend the sequence to the leveal of $n + 1$. Now let us prove such diagram really exists.

First we describe $\mathrm{Cone}(\Sigma X)$ and $\Sigma\mathrm{Cone}(X)$ concretely.

$$\mathrm{Cone}(\Sigma X) = \frac{X \times I \times I}{(\{x_0\} \times I \times I) \cup (X \times \partial I \times I) \cup (X \times I \times 0)}$$
$$\Sigma\mathrm{Cone}(X) = \frac{X \times I \times I}{(\{x_0\} \times I \times I) \cup (X \times 0 \times I) \cup (X \times I \times \partial I)}$$

They are homoemorphic via μ. This homeomorphism is denoted by $\overline{\mathcal{X}}$.



where $\Sigma i_1^X = \overline{\mathcal{X}} \circ i_1^{\Sigma X} : (x, s) \mapsto (x, s, 1) \mapsto (x, 1, s)$. Since Σ is left adjoint to Ω hence preserving colimits especially pushouts. Then by the uniqueness of pushouts, we see $\mathcal{X}$ is a homeomorphism. And we can use the universal property of pushouts to check $\mu \circ p(\Sigma f) = \Sigma p(f) \circ \mathcal{X}$. □

1.5 Cofibrations

Definition 1.44. *For any map $f : X \to Y$, the **mapping cylinder** of f is defined to be the following pushout*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & Y \arrow[d, hook, "f_0"] \\ X \times I \arrow[r] & \mathrm{Cyl}(f) \end{tikzcd}$$

then $\mathrm{Cyl}(f) = \frac{(X\times I)\coprod Y}{(x,0)\sim f(x)}$.

Remark 1.45. Consider the map $r : \mathrm{Cyl}(f) \to Y$ such that $y \mapsto y,\ (x,s) \mapsto f(x)$. $r \circ f_0 = \mathrm{id}_Y$. Then r is a retraction. Actually $f_0 : Y \hookrightarrow \mathrm{Cyl}(f)$ is a strong deformation retract. For $f_0 \circ r : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f),\ y \mapsto y,\ (x,s) \mapsto f(x) = (x,0)$, we define the homotopy

$$H : \mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(f),\ (x,s,t) \mapsto (x,st),\ (y,t) \mapsto y$$

then $H : f_0 \circ r \simeq \mathrm{id}_{\mathrm{Cyl}(f)}$. And we will have a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i_1"'] \arrow[rr, bend left, "f"] & \mathrm{Cyl}(f) \arrow[r, two heads, "\sim", "r"'] & Y \end{tikzcd}$$

where r is a homotopy equivalence.

Definition 1.46. *Given a map $i : A \to X$ we say it has the **homotopy extension property** (HEP) or call it **cofibration** if for any space Z and maps $g : X \to Z$, $h : A \times I \to Z$ satisfying $g \circ i = h \circ i_0^A$ there exists $H : X \times I \to Z$ such that $H \circ i_0^X = g,\ H \circ (i \times \mathrm{id}) = h$.*

$$\begin{tikzcd} A \arrow[r, "i_0^A"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "h"] & \\ X \arrow[r, "i_0^X"] \arrow[drr, bend right, "g"'] & X \times I \arrow[dr, dotted, "H", "\exists"'] & \\ & & Z \end{tikzcd}$$

Remark 1.47. Consider the mapping cylinder diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, hook, "i_0^A"'] & X \arrow[d, hook, "i_0"] \\ A \times I \arrow[r] & (A \times I) \cup_i (X \times 0) = \mathrm{Cyl}(i) \end{tikzcd}$$

The diagram in Definition 1.46 is equivalent to the following diagram

$$\begin{array}{ccc} (A \times I) \cup_i (X \times 0) & \xrightarrow{(h,g)} & Z \\ {\scriptstyle (i\times \mathrm{id}, i_0^X)}\downarrow & \overset{\exists\, H}{\nearrow} & \downarrow \\ X \times I & \longrightarrow & * \end{array}$$

Hence HEP is just a special case of lifting property. More precisely, using the adjunction induced by $- \times I$ and $(-)^I$ in **Top**, we see the lifting problem is also equivalent to the following one

$$\begin{array}{ccc} A & \xrightarrow{\tilde{h}} & Z^I \\ {\scriptstyle i}\downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow{\scriptstyle e_0} \\ X & \xrightarrow[g]{} & Z \end{array}$$

Hence cofibrations are just those maps having the left lifting property with respect to e_0. The class is just $l(e_0)$ where this notation comes from the Chapter of Homotopical Algebra. Then according to Theorem 2.10 the class of cofibrations is closed under retracts, pushouts, coproducts and transfinite compositions.

Remark 1.48. In the diagram above (second diagram in Remark 1.47), we let $(h, g) = id$ and $Z = \mathrm{Cyl}(i)$. This means there will exist a map $r : X \times I \to \mathrm{Cyl}(i)$ such that $r \circ j = \mathrm{id}_{\mathrm{Cyl}(i)}$ where $j = (i \times \mathrm{id}, i_0^X)$. Then especially j is an injection. Actually for any map $i : A \to X$, it's a cofibration iff j admits a retraction r. In this case, H will be $(h, g) \circ r$. Consider the following factorization

$$A \overset{i_1^A}{\hookrightarrow} \mathrm{Cyl}(i) \overset{j}{\hookrightarrow} X \times I, \qquad j \circ i_1^A = i \times 1$$

Note that i_1^A is actually an injection since $\mathrm{Cyl}(i) = \frac{(A\times I)\coprod X}{(a,0)\sim i(a)}$. This means the cofibration i is an injection. Later we will see i is even an embedding.

Lemma 1.49. *Given an equalizer in* **Top**

$$X \longrightarrow Y \underset{g}{\overset{f}{\rightrightarrows}} Z$$

where Z is Hausdorff, then X is a closed subspace of Y.

Proof. $X = \{y \in Y | f(y) = g(y)\}$ with subspace topology. To prove $X \subseteq Y$ is closed we need to prove $Y - X$ is open. Given $y \in Y - X$, $f(y) \neq g(y)$. Since Z is Hausdorff, there are disjoint open subsets U, V of Z containing $f(y), g(y)$ respectively. $y \in f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$. And $f^{-1}(U) \cap g^{-1}(V) \cap X = \emptyset$ otherwise there will exist $x \in X$, $f(x) = g(x) \in U \cap V$. □

Proposition 1.50. *The cofibration $i : A \to X$ is actually an embedding. If X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{h}} & \mathrm{Cyl}(i)^I \\ {\scriptstyle i}\downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow{\scriptstyle e_0} \\ X & \xrightarrow[i_0]{} & \mathrm{Cyl}(i) \end{array}$$

where $\tilde{h}$ is induced from $h : A \times I \to \mathrm{Cyl}(i)$ the pushout of $i : A \to X$ (see Remark 1.47). Let $u = \tilde{h}(-)(\frac{1}{2}) : A \to \mathrm{Cyl}(i)$, $v = \widetilde{H}(-)(\frac{1}{2}) : X \to \mathrm{Cyl}(i)$.

Assume $i' : A \to i(A)$, $u' : A \to u(A)$, $v' : i(A) \to u(A)$. Then $v' \circ i' = u' : A \to u(A)$. And $u(a) = \tilde{h}(a)(\frac{1}{2}) = h(a, \frac{1}{2}) = (a, \frac{1}{2})$ in $\mathrm{Cyl}(i) = \frac{(A\times I)\coprod X}{(a,0)\sim i(a)}$. This is an embedding. Therefore u' is a homeomorphism. ${u'}^{-1} \circ v' \circ i' = \mathrm{id}_A$. i' has a continuous inverse. Since in Remark 1.48, we have proved i is injective which means i' is a bijection. We conclude i' is a homeomorphism.

Next we assume X is Hausdorff and then $X \times I$ is Hausdorff as well.

$$X \times I \xrightarrow{r} \mathrm{Cyl}(i) \overset{j}{\hookrightarrow} X \times I, \qquad j \circ r \text{ vs. } \mathrm{id}$$

Consider the equalizer of $\mathrm{id}_{X\times I}$ and $j \circ r$ where symbols j and r come from Remark 1.48. This equalizer is just $j(\mathrm{Cyl}(i)) = (i(A) \times I) \cup (X \times 0)$ which is closed in $X \times I$ according to Lemma 1.49. We use this to prove $i(A)$ is closed in X.

Given $x \notin i(A)$, $(x, \frac{1}{2}) \notin i(A) \times I$ and then $(x, \frac{1}{2}) \notin j(\mathrm{Cyl}(i))$. There will exists open subsets W_1, W_2 such that $(x, \frac{1}{2}) \in W_1 \times W_2 \subseteq X \times I$ and $(W_1 \times W_2) \cap j(\mathrm{Cyl}(i)) = \emptyset$. The latter means $(W_1 \times W_2) \cap (i(A) \times I) = (W_1 \cap i(A)) \times W_2 = \emptyset \Rightarrow W_1 \cap i(A) = \emptyset$. This proves $X - i(A)$ is open. □

Remark 1.51. For a cofibration $i : A \hookrightarrow X$, whether $i(A) \subseteq X$ is closed or not is important. If i is a closed embedding, then $\mathrm{Cyl}(i) = (A \times I) \cup_i (X \times 0) = (A \times 0) \cup (X \times 0)$. The topology induced by pushouts coincides with the subspace topology. Since for any maps $g : X \to Z$ and $h : A \times I \to Z$ with $g|A = h|A \times 0$, they define a continuous map $(h, g) : (A\times I)\cup(X\times 0) \to Z$. Note in this case $A\times I$ and $X\times 0$ are closed in $(A\times I)\cup(X\times 0)$ under the subspace topology. Since the continuous map glued along finite closed subsets is continuous (Lemma 1.52), (h, g) will be continuous. Hence $(A\times I)\cup(X\times 0)$ satisfies the universal property of pushouts. Then $(A \times I) \cup (X \times 0) = \mathrm{Cyl}(i)$. According to Remark 1.48, to know a closed embedding $i$is whether a cofibration, we only need to check if there exists a retraction for the natural embedding $(A \times I) \cup (X \times 0) \hookrightarrow X \times I$. And finally we obtain the Corollary 1.53

Lemma 1.52. *$X = \cup_1^n X_i$ where X_i's are closed in X. Given maps $f_i : X_i \to Y$ satisfying $f_i|X_i \cap X_j = f_j|X_i \cap X_j$, then this defines a unique continuous map $f : X \to Y$.*

Proof. The uniqueness is obvious and we only need to prove it's continuous. For any closed subset $Z \subseteq Y$, $f^{-1}(Z) = \cup_1^n f_i^{-1}(Z)$ where $f_i^{-1}(Z)$ is closed in X_i hence closed in X. Then $f^{-1}(Z)$ is closed. □

Corollary 1.53.

(1) *If* $i : A \to X$ *is a cofibration, then the natural embedding* $(A \times I) \cup (X \times 0) \hookrightarrow X \times I$ *has a retraction. It's actually a strong deformation retract.*

(2) *If* $A \subseteq X$ *is closed and there is a retraction for* $(A \times I) \cup (X \times 0) \hookrightarrow X \times I$*, then* $A \hookrightarrow X$ *is a cofibration.*

Proof. (1). The existence of such retraction is the same as that in Remark 1.48. Note there is a natural map $(A\times I)\cup_i (X\times 0) \to (A\times I)\cup(X\times 0)$, whose images in $X\times I$ coincide. Then we only need to prove this retraction $r : X\times I \to (A\times I)\cup(X\times 0)$ is a strong deformation retract, which is equivalent to find $H : X\times I\times I \to X\times I,\ j\circ r \simeq \mathrm{id}_{X\times I}\,\mathrm{rel}(A\times I)\cup(X\times 0)$.

$$\begin{array}{ccc}
(X\times I\times 0)\cup(X\times I\times 1)\cup(A\times I\times I)\cup(X\times 0\times I) & \xrightarrow{(j\circ r,\mathrm{id}_{X\times I},pr_{A\times I},pr_{X\times 0})} & X\times I\times I \\
\downarrow & \overset{\exists\, H}{\nearrow} & \\
X\times I\times I & &
\end{array}$$

H is defined to be

$$(x,s,t) \mapsto \big(pr_1 r(x,(1-t)s),(1-t)pr_2 r(x,s)+st\big)$$

(2). See Remark 1.51. ∎

From this corollary, we see $|\partial\Delta^n| \hookrightarrow |\Delta^n|$ is a cofibration.

Theorem 1.54. *In Remark 1.45, in such factorization of* $f : X \to Y$*,* r *is a homotopy equivalence and* i_1 *is a cofibration.*

Proof. It only remains to prove i_1 is a cofibration. $i_1 : X \to \mathrm{Cyl}(f) = \frac{(X\times I)\coprod Y}{(x,0)\sim f(x)},\ x\mapsto(x,1)$ is a closed embedding. Then from the Corollary 1.53 (2) it's enough to prove $(X\times 1\times I)\cup(\mathrm{Cyl}(f)\times 0) \hookrightarrow \mathrm{Cyl}(f)\times I$ admits a retraction.

Define $r : \mathrm{Cyl}(f)\times I \to (X\times 1\times I)\cup(\mathrm{Cyl}(f)\times 0)$

$$(y,t)\mapsto(y,0),\ (x,s,t)\mapsto \begin{cases}(x,\frac{s}{1-t},0), & 0\le s\le 1-t\\ (x,1,,s-1+t), & 1-t\le s\le 1\end{cases}$$

∎

Finally in this section we want to talk about the concept of the *cofiber homotopy relation*. Given a fixed topological space A now we work on the category of $A/\mathbf{Top}$, in which objects are morphisms $i : A \to X$ and morphisms between $i,\ j$ are the following commutative diagram

$$\begin{array}{ccccc}
 & & A & & \\
 & {\scriptstyle i}\swarrow & & \searrow{\scriptstyle j} & \\
X & & \xrightarrow[f]{} & & Y
\end{array}$$

There is also a homotopy relation in $\mathrm{Hom}_{A/\mathbf{Top}}(i, j)$. Given a usual homotopy $H : X \times I \to Y,\ f \simeq g$, we say it's under A if for any $t \in I$, $H(-, t) \circ i = j$, which is written as $H : f \simeq g \text{ rel } A$. Obviously the homotopy relation under A is an equivalence relation and we have the concept of homotopy equivalence in $A/\mathbf{Top}$. If moreover i and j are cofibrations, such homotopy equivalence is called the *cofiber homotopy equivalence*. However in this case cofiber homotopy relations and usual homotopy equivalences are equivalent. We will prove this theorem and the proof comes from [May99] Chapter 6 Section 5.

Theorem 1.55. *Let $i : A \to X$ and $j : A \to Y$ be cofibrations and $f : X \to Y$ is a morphism in $A/\mathbf{Top}$. If f is a usual homotopy equivalence, then it's a cofiber homotopy equivalence.*

Proof. We should find a map $g : Y \to X$ in $A/\mathbf{Top}$ such that $g \circ f \simeq \mathrm{id}_X \text{ rel } A$. Then since f is a homotopy equivalence, g will be a homotopy equivalence as well. Then there will also exists a map $f' : X \to Y$ in $A/\mathbf{Top}$ such that $f' \circ g \simeq \mathrm{id}_Y \text{ rel } A$. Then g has a left homotopy inverse and a right homotopy inverse under A, both of which will be homotopy equivalent under A and this can be checked in the homotopy category of $A/\mathbf{Top}$. Therefore to prove this theorem it's enough to find such g.

Since f is a homotopy equivalence, there is a map $g'' : Y \to X$ such that $g'' \circ f \simeq \mathrm{id}_X \Rightarrow\ g'' \circ f \circ i = g'' \circ j \simeq i$. Given a homotopy $H : A \times I \to X,\ g'' \circ j \simeq i$, since j is a cofibration, the following lifting problem has a solution $K : Y \times I \to X$

$$\begin{array}{ccc} (A \times I) \cup_j (Y \times 0) & \xrightarrow{(H,\, g'')} & X \\ {\scriptstyle (j\times \mathrm{id}, i_0^Y)} \big\downarrow & \overset{K}{\nearrow} & \\ Y \times I & & \end{array}$$

Let $g' = K(-, 1)$. We see $K : g'' \simeq g'$ and $g' \circ j = H(-, 1) = i$. This means g' is a map in $A/\mathbf{Top}$ and it's the homotopy inverse of f in the usual sense. Now we only need to prove $g' \circ f : X \to X$ admits a left homotopy inverse $e : X \to X$ under A. Then we can define $g = e \circ g'$. Hence replacing our original map f by $g' \circ f$, we can assume our f satisfying $f \circ i = i$ and $f \simeq \mathrm{id}_X$. And we prove such f has a left homotopy inverse under A.

Choose a homotopy $h : X \times I \to X,\ f \simeq \mathrm{id}_X$. Since $i : A \to X$ is a cofibration, the following diagram admits a lifting $k : X \times I \to X,\ \mathrm{id}_X \simeq k_1 = e$.

$$\begin{array}{ccc} (A \times I) \cup_i (X \times 0) & \xrightarrow{(h\circ(i\times \mathrm{id}),\, \mathrm{id}_X)} & X \\ {\scriptstyle (i\times \mathrm{id}, i_0^X)} \big\downarrow & \overset{k}{\nearrow} & \\ X \times I & & \end{array}$$

From the diagram we see $e \circ i = h \circ (i \times \mathrm{id})|A \times 1 = \mathrm{id}_X \circ i = i$. Then e is in $A/\mathbf{Top}$. $e,\ f$ are both homotopic to id_X, and therefore there is a homotopy $J : X \times I \to X,\ e \circ f \simeq \mathrm{id}_X$. More specifically

$$J(x, s) = \begin{cases} k(f(x), 1 - 2s), & s \leq \frac{1}{2} \\ h(x, 2s - 1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Now we consider the following complicated lifting problem

$$\begin{array}{ccc} (A \times I \times I) \cup_{i\times \mathrm{id}} (X \times I \times 0) & \xrightarrow{(J',\, J)} & X \\ \big\downarrow & \overset{L}{\nearrow} & \\ X \times I \times I & & \end{array}$$

Before describing J' in this diagram, we explain why $i \times \mathrm{id} : A \times I \to X \times I$ is a cifibration. Look at the third diagram in Remark 1.47. Since I is locally compact Hausdorff, $Z^{I\times I} \approx (Z^I)^I$. Replacing Z in that diagram by Z^I, we see $i \times \mathrm{id} : A \times I \to X \times I$ is a cofibration as well.

$$J'(a, s, t) = \begin{cases} k(i(a), 1 - 2s(1-t)), & s \leq \frac{1}{2} \\ h(i(a), 1 - 2(1-s)(1-t)), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

The lifting $L : X \times I \times I \to X$ gives a homotopy

$$e \circ f = J_0 = L_{0,0} \simeq L_{0,1} \simeq L_{1,1} \simeq L_{1,0} = J_1 = \mathrm{id}_X \text{ rel } A$$

Note that in the sequence of homotopy relations above, $L_{0,0} \simeq L_{1,0}$ rel A follows from $L_{0,0} \simeq L_{0,1}$ rel A, $L_{0,1} \simeq L_{1,1}$ rel A and $L_{1,1} \simeq L_{1,0}$ rel A but not from J directly. □

1.6 Fibrations

Dual to the diagram

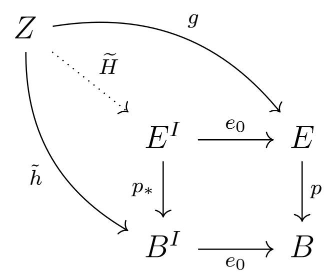
$$\begin{array}{ccc} A & \xrightarrow{\tilde{h}} & Z^I \\ {\scriptstyle i}\big\downarrow & \overset{\widetilde{H}}{\nearrow} & \big\downarrow{\scriptstyle e_0} \\ X & \xrightarrow[g]{} & Z \end{array}$$

in the definition of cofibrations (see Remark 1.47), we can define the *fibration (Hurewicz)* as follows.

Definition 1.56. *A map $p : E \to B$ is a* ***(Hurewicz) fibration*** *if for any spaces Z and maps $g : Z \to E$, $h : Z \times I \to B$ satisfying $h \circ i_0^Z = p \circ g$ there exists $H : Z \times I \to E$ such that $H \circ i_0^Z = g, p \circ H = h$. This means the following diagram is commutative*

$$\begin{array}{ccc} Z & \xrightarrow{g} & E \\ {\scriptstyle i_0^Z}\big\downarrow & \overset{H}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow[h]{} & B \end{array}$$

or equivalently



Remark 1.57. This property is called the *homotopy lifting property* (HLP). From the section of Homotopical Algebra, we know the class of fibrations is just $r(i_0^Z)$. Then they are closed under retracts, pullbacks, products and compositions.

Example 1.58. In the Definition 1.56, if $B = *$, then we can let $H = g \circ pr_1$. Hence any $E \to *$ is a fibration. Then for any space E, $B \times E \to B$ is a fibration since it's a product of fibrations.

Similar to Remark 1.45, for any continuous map $f : X \to Y$ we can replace it by a fibration homotopically.

Definition 1.59. *For any map $f : X \to Y$, its* ***mapping path space*** *is defined to be the following pullback*

$$\begin{array}{ccc} \mathrm{Path}_0(f) & \longrightarrow & Y^I \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle e_0} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and

$$\mathrm{Path}_0(f) = \{(x, \varphi) | \varphi : I \to Y,\ \varphi(0) = f(x)\}$$

Remark 1.60. Define $i : X \hookrightarrow \mathrm{Path}_0(f),\ x \mapsto (x, \mathrm{cst}_{f(x)})$ where $\mathrm{cst}_{f(x)}$ denotes the constant map at $f(x)$. Then we have the following factorization

$$X \underset{i}{\overset{\sim}{\hookrightarrow}} \mathrm{Path}_0(f) \underset{p}{\twoheadrightarrow} Y, \qquad p \circ i = f$$

where

$$p : \mathrm{Path}_0(f) \longrightarrow Y^I \xrightarrow{e_1} Y \ , \ (x, \varphi) \mapsto \varphi(1)$$

Next we prove $i : X \hookrightarrow \mathrm{Path}_0(f)$ is a strong deformation retract and p is a fibration.

Proof. Define the retraction $r : \mathrm{Path}_0(f) \to X,\ (x, \varphi) \mapsto x$. Then it's easy to check $r \circ i = \mathrm{id}_X$.

$$i \circ r : \mathrm{Path}_0(f) \to \mathrm{Path}_0(f),\ (x, \varphi) \mapsto (x, \mathrm{cst}_{f(x)})$$

We define the homotopy as

$$H : \mathrm{Path}_0(f) \times I \to \mathrm{Path}(f),\ (x, \varphi, t) \mapsto (x, s \mapsto \varphi(st))$$

Since the map

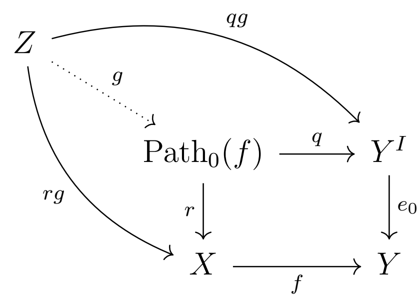
$$Y^I \times I \times I \xrightarrow{(g,s,t)\mapsto(g,st)} Y^I \times I \xrightarrow{(g,st)\mapsto g(st)} Y$$

is continuous, $H' : Y^I \times I \xrightarrow{(g,s)\mapsto(t\mapsto g(st))} Y^I$ is continuous. Hence H is continuous and it's then obvious to check $H : i \circ r \simeq \mathrm{id}_{\mathrm{Path}_0(f)}$ rel $i(X)$.

It remains to prove p is a fibration. Given a lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{g} & \mathrm{Path}_0(f) \\ {\scriptstyle i_0^Z}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow[h]{} & Y \end{array}$$

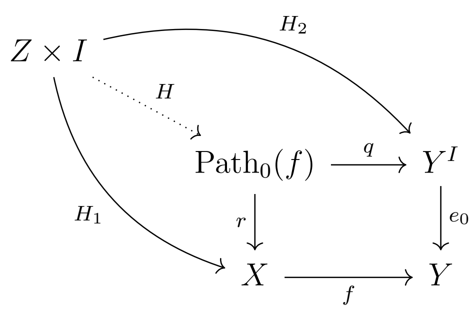
we want to find a map $H : Z \to \mathrm{Path}_0(f)$ filling in the diagram. The map $g : Z \to \mathrm{Path}_0(f)$ is actually equivalent to the following diagram



$g(z) = (rg(z), qg(z))$, $pg(z) = qg(z)(1) = h(z, 0)$. To find such a lifting H

$$\begin{array}{ccc} z & \longmapsto & (rg(z), qg(z)) \\ \big\downarrow & {\scriptstyle H} & \big\downarrow \\ (z,0) & \longmapsto & h(z,0) = qg(z)(1) \end{array}$$

we need to solve



where $H_1(z, s) = rg(z)$. For H_2, its adjuncion is $H_2' : Z \times I \times I \to Y$. For $(z, s, t) \in Z \times I \times I$ when $s = 0$ it should be $qg(z)(t)$ and when $t = 1$ it should be $h(z, s)$. Therefore we can define H_2' as follows

$$H_2' : Z \times I \times I \to Y,\ (z, s, t) \mapsto \begin{cases} qg(z)((1 + s)t), & t \le \frac{1}{1+s} \\ h(z, (1 + s)t - 1), & \frac{1}{1+s} \le t \le 1 \end{cases}$$

If $t=0$, $H_2'(z,s,0) = qg(z)(0) = f(rg(z))$. This defines $H : Z \times I \to \mathrm{Path}_0(f)$. $\square$

In the following we will give some examples of fibrations and the famous one is *covering spaces*, but we won't need them here. We plan to partially complete the unfinished work in Example 1.23 and prove $(e_0, e_1) : X \to X \times X$ is a fibration. To do this we need a helpful theorem.

Theorem 1.61.

(1) If $i : A \hookrightarrow X$ is a cofibration between locally compact Hausdorff spaces, then $i^* : Z^X \to Z^A$ is a fibration for all spaces Z.

(2) If $p : E \to B$ is a fibration, then $p_* : E^Z \to B^Z$is a fibration for all locally compact Hausdorff spaces Z.

Proof. (1). The following lifting problem

$$\begin{array}{ccc} B & \longrightarrow & Z^X \\ {\scriptstyle i_0^B}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle i^*} \\ B \times I & \longrightarrow & Z^A \end{array}$$

is equivalent to a complicated one

$$\begin{array}{ccccc} B \times A \times 0 & \xrightarrow{\mathrm{id}_B \times i_0^A = i_0^{B\times A}} & B \times A \times I & & Z \\ {\scriptstyle \mathrm{id}_B \times i}\downarrow & & \downarrow & \overset{\exists !}{\nearrow} & \uparrow{\scriptstyle \widetilde{H}} \\ B \times X \times 0 & \longrightarrow & (B \times X \times 0) \cup_{\mathrm{id}_B \times i} (B \times A \times I) = \mathrm{Cyl}(\mathrm{id}_B \times i) & & \\ & & & \searrow & \\ & & & & B \times X \times I \end{array}$$

We explain why they are equivalent. At first we have a commuative diagram

$$\begin{array}{ccc} B \times A & \longrightarrow & B \times X \\ {\scriptstyle i_0^B \times \mathrm{id}_A}\downarrow & & \downarrow \\ B \times I \times A & \longrightarrow & Z \end{array}$$

which induces a unique morphism $\mathrm{Cyl}(\mathrm{id}_B \times i) \to Z$. Then the equivalence will be easily checked at the level of $B \times A \times I$ and $B \times X \times 0$. Now it's enough to prove $\mathrm{id}_B \times i$ is a cofibration. In the proof of Theorem 1.55, we have proved $i \times \mathrm{id}_I$ is a cofibration if i is a cofibration. But that proof is not valid here since we do not suppose B is Hausdorff.

From the Proposition 1.50, since X is Hausdorff $i(A)$ is closed in X. Hence $B \times A$ is closed in $B \times X$. Consider the inclusion $(B \times X \times 0) \cup (B \times A \times I) \subseteq B \times X \times I$. Since i is a cofibration, the inclusion $(X \times 0) \cup (A \times I) \subseteq X \times I$ admits a retraction $r : X \times I \to (X \times 0) \cup (A \times I)$. Then $\mathrm{id}_B \times r : B \times X \times I \to (B \times X \times 0) \cup (B \times A \times I)$ will be the retraction, which means $\mathrm{id}_B \times i$ is a cofibration from the Corollary 1.53 (2).

(2). The following two lifting problems are equivalent

$$\begin{array}{ccc} X & \longrightarrow & E^Z \\ {\scriptstyle i_0^X}\downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & B^Z \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} X \times Z & \longrightarrow & E \\ {\scriptstyle i_0^{X\times Z}}\downarrow & \nearrow & \downarrow {\scriptstyle p} \\ X \times Z \times I & \longrightarrow & B \end{array}$$

□

Example 1.62.

(1) Consider the closed embedding $i : \partial I \hookrightarrow I$. Since $(I \times 0) \cup (\partial I \times I)$ is a strong deformation retract of $I \times I$, i is a cofibration between locally compact Hausdorff spaces. Hence for any space X,

$$(e_0, e_1) = i^* : X^I \to X^{\partial I} \approx X \times X,\ \varphi \mapsto (\varphi(0), \varphi(1))$$

is a fibration.

(2) Given a closed embedding $i_t : * \hookrightarrow I,\ * \mapsto t$, which is a cofibration since $I \times 0) \cup (t \times I)$ is a strong deformation retract of $I \times I$, where the retraction is defined to be the projection along the parallel line of $y = -\frac{1}{t}x + 1$ from left to right in $[0, t] \times I$ and along the parallel line of $y = \frac{1}{1-t}x - \frac{t}{1-t}$ from right to left in $[t, 1] \times I$. Then

$$e_t = i_t^* : X^I \to X^* = X,\ \varphi \mapsto \varphi(t)$$

is a fibration.

Like the previous section, at the end of this section we talk about *fiber homotopy equivalence* on the category of $\mathbf{Top}/B$. All concepts here are dual and we will not write them again. Dual to Theorem 1.55, we have the following one.

Theorem 1.63. *Let $p : X \to B$ and $q : Y \to B$ be fibrations and $f : X \to Y$ is a morphism in $\mathbf{Top}/B$. If f is a usual homotopy equivalence, then it's a fiber homotopy equivalence.*

1.7 Strøm's Model Category Structure on Top

Recall the definition of model categories (see Definition 2.18).

Theorem 1.64. *With cofibrations being closed cofibrations (see Definition 1.46), which means the image of it is closed, weak equivalences being homotopy equivalences and fibrations being Hurewicz fibrations,* **Top** *is a model category. This model category structure is called Hurewicz model category structure or Strøm's model category structure.*

Our task of this section is to prove the theorem above, which is due to Strøm. At first we prove the (M4) about lifting properties.

Lemma 1.65. *Let $p : E \to B$ be a fibration and A is a deformation retract of X. Moreover if there is a map $\varphi : X \to I$ such that $A = \varphi^{-1}(0)$, then any lifting problem*

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & B \end{array}$$

can be solved.

Proof. Let $r : X \to A$ be the retraction and $H : X \times I \to X,\ ir \simeq \mathrm{id}_X$. Define

$$H' : X \times I \to X,\ (x, s) \mapsto \begin{cases} H(x, \frac{s}{\varphi(x)}), & s < \varphi(x) \\ H(x, 1), & s \geq \varphi(x) \end{cases}$$

Then we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{fr} & E \\ {\scriptstyle i_0^X}\big\downarrow & \overset{F}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[gH']{} & B \end{array}$$

When $s = 0$,

$$H'(x, 0) = \begin{cases} H(x, 0) = ir(x), & x \notin A \\ H(x, 1) = x, & x \in A \end{cases}$$

hence $H'(x, 0) = ir(x)$. $gH'i_0^X = pfr = gir$. Let $h(x) = F(x, \varphi(x))$. Such $h : X \to E$ is the lifing we want. $ph(x) = pF(x, \varphi(x)) = gH'(x, \varphi(x)) = g(x)$ and $hi(a) = F(a, \varphi(a)) = F(a, 0) = fr(a) = f(a)$. □

Proposition 1.66. *If $p : E \to B$ is a fibration and $A \hookrightarrow X$ is a closed cofibration, then any lifting problem*

$$\begin{array}{ccc} (X \times 0) \cup (A \times I) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X \times I & \longrightarrow & B \end{array}$$

can be solved.

Proof. Let $r : X \times I \to (X \times 0) \cup (A \times I)$ be the retraction. We define the map

$$\varphi : X \to I,\ x \mapsto \sup_{t \in I} |t - pr_2 r(x, t)|$$

and then $\varphi(x) = 0$ iff $\forall t \in I,\ t = pr_2 r(x, t)$. If $a \in A$, then $r(a, t) = (a, t) \Rightarrow t = pr_2 r(a, t)$. Conversely, we suppose $x \notin A$. Consider the path

$$\mu : x \times I \hookrightarrow X \times I \xrightarrow{r} (X \times 0) \cup (A \times I) \xrightarrow{pr_1} X$$

$\mu(0) = x \notin A$. Since A is closed in X, $\mu^{-1}(A^c)$ is open in I containing 0. Then there will be an interval of the form $[0, s) \subseteq \mu^{-1}(A^c)$ with $s > 0$. Hence the image of $r|x \times [0, s)$ lies in $X \times 0$ and for all $t \in (0, s)$, $pr_2 r(x, t) = 0 \neq t$. This proves $\varphi^{-1}(0) = A$.

Next define $\psi : X \times I \to I,\ (x, t) \mapsto t\varphi(x)$. We see $\psi^{-1}(0) = (X \times 0) \cup (A \times I)$. Then applying Lemma 1.65, we conclude the lifting problem can be solved. □

Proposition 1.67. *Suppose $i : A \hookrightarrow X$ is a closed cofibration and $p : E \to B$ is both a fibration and a homotopy equivalence (trivial fibration). Then any lifting problem*

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ {\scriptstyle i}\big\downarrow & \overset{\bar{f}}{\nearrow} & \big\downarrow \\ X & \xrightarrow[f']{} & B \end{array}$$

can be solved.

Proof. Since p is a homotopy equivalence, there is some $s : B \to E$ satisfying $p \circ s \simeq \mathrm{id}_B,\ s \circ p \simeq \mathrm{id}_E$. Considering the lifting problem

$$\begin{array}{ccc} B & \xrightarrow{s} & E \\ {\scriptstyle i_0^B}\big\downarrow & \overset{H}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B \times I & \xrightarrow[p \circ s \simeq \mathrm{id}_B]{} & B \end{array}$$

Let $s' = H(-, 1)$. We see $p \circ s' = \mathrm{id}_B$ and $H : s \simeq s'$, which means s' is the homotopy inverse of p as well. Replacing s by s' we can just assume $p \circ s = \mathrm{id}_B$.

$$\begin{array}{ccc} & & E \\ & \overset{sp}{\nearrow} & \big\downarrow{\scriptstyle p} \\ E & \xrightarrow[p]{} & B \end{array}$$

Then sp is a morphism in $\mathbf{Top}/B$. Since sp is a homotopy equivalence over B with p being a fibration, by Theorem 1.63, we see sp is a fiber homotopy equivalence. Hence we can assume there is some map $s' : E \to E$ over B and a fiber homotopy $F : E \times I \to E,\ s' \circ s \circ p \simeq \mathrm{id}_E$ which means it satisfies $pF(e, t) = p(e)$.

According to Proposition 1.66, the following lifting problem admits a solution

$$\begin{array}{ccc} (X \times 0) \cup (A \times I) & \xrightarrow{F''} & E \\ {\scriptstyle j}\big\downarrow & \overset{\overline{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F']{} & B \end{array}$$

where

$$F'' : \begin{cases} (x, 0) \mapsto s' s f'(x) \\ (a, t) \mapsto F(f''(a), t) \end{cases}$$

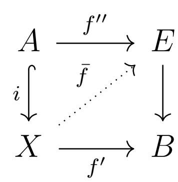
and $F'(x,t) = f'(x)$. Note that on $A \times 0$, $F(f''(a), 0) = s'spf''(a) = s'sf'i(a) = s'sf'(a)$, which means F'' is well defined and is continuous.

On $X \times 0$, $pF''(x, 0) = ps'sf'(x) = f(x)$ and on $A \times I$, $pF''(a,t) = pF(f''(a),t) = pf''(a) = f'(a)$. Hence $F' \circ j = p \circ F''$. Define $\bar{f}(x) = \overline{F}(x,1)$ and then

$$\bar{f}i(a) = \overline{F}(a,1) = F''(a,1) = F(f''(a),1) = f''(a)$$

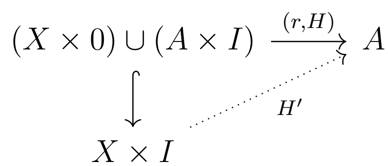
and $p\bar{f}(x) = p\overline{F}(x,1) = F'(x,1) = f'(x)$. $\square$

Proposition 1.68. *Assume $i : A \hookrightarrow X$ is both a closed cofibration and a homotopy equivalence, and $p : E \to B$ is a fibration. Then any lifting problem*



can be solved.

Proof. $r : X \to X$ is the homotopy inverse of i. Let $H : A \times I \to A,\ r \circ i \simeq \mathrm{id}_A$.

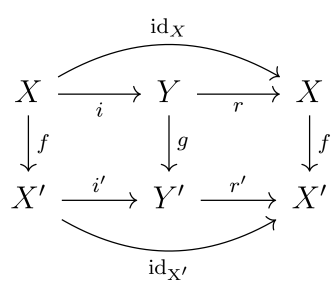


Let $r'(x) = H'(x,1) \Rightarrow H' : r \simeq r'$ and $r'(a) = H(a,1) = a$. $ir \simeq ir' \simeq \mathrm{id}_X$. Hence $r' : X \to A$ is the deformation retraction of $A \subseteq X$. Combining Lemma 1.65 with the first part of the proof of Proposition 1.66, we conclude the lifting $\bar{f}$ exists. $\square$

Proposition 1.67 and 1.68 have proved the (M4) of model categories. (M2) is clear in the homotopy category **HoTop**. Next we prove (M3) is true.

Proposition 1.69. *Closed cofibrations, Hurewicz fibrations and homotopy equivalences are closed under retraction.*

Proof. It's enough to only prove the retraction of a closed cofibration is closed as well.



Suppose in the diagram above g is a closed cofibration. $\mathrm{im}\ g$ is closed $\Rightarrow$ $i'^{-1}(\mathrm{im}\ g)$ is closed. Since $i'f = gi$ and i' is injective, $\mathrm{im}\ f \subseteq i'^{-1}(gi(X)) \subseteq i'^{-1}(\mathrm{im}\ g)$. On the other hand, if $x' \in X'$ such that $i'(x') \in \mathrm{im}\ g$. Then there is some $y \in Y$, $i'(x') = g(y) \Rightarrow r'i'(x') = x' = r'g(y) = fr(y)$. Hence $x' \in \mathrm{im}\ f$. We conclude $\mathrm{im}\ f = i'^{-1}(\mathrm{im}\ g)$ is closed. $\square$

Proposition 1.70. *For any continuous map $f : X \to Y$, $f = pi = qj$ where p is a fibration, i is both a closed cofibration and a homotopy equivalence, q is both a fibration and a homotopy equivalence and j is a closed cofibration.*

We don't plan to prove this factorization theorem here since the method Strøm uses (see [Str72] Proposition 2) is limited and it's difficult to apply it to any other situations, which is different from Quillen's *small object argument*. The limitation of Strøm's method actually lies in the fact that Strøm's model category structure on **Top** is not *cofibrantly generated*. A genelization of Strøm's idea can be found in [BRi12] which can be applied to other model categories not necessarily cofibrantly generated.

1.8 Quillen's Model Category Structure on Top

Theorem 1.71. *With fibrations being Serre fibrations and weak equivalences being weak homotopy equivalences,* **Top** *is a model category. This model category structure is called Quillen model category structure.*

Before explaining what the *Serre fibration* is, we explain why cofibrations do not appear in Theorem 1.71 (see Corollary 2.29). In a model category $\mathcal{M}$, cofibrations are just those having the left lifting property with respect to all trivial fibrations. Hence if we have defined fibrations and weak equivalences, we can just define cofibrations to be those maps having this lifting property. In this section we prove Theorem 1.71 via the standard *Quillen's small object argument* using the characterizations of fibrations and trivial fibrations. Note that in this section, the word "fibration" always means the Serre fibration.

Definition 1.72. *Let $p : X \to B$ be a continuous map. It's a* ***Serre fibration*** *if for any integer $n \geq 1,\ 0 \leq k \leq n$ the following lifting problem*

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ |\Delta^n| & \longrightarrow & B \end{array}$$

admits a solution.

Cofibrations are defined to be those maps having the LLP wrt all trivial Serre fibrations. Note that for simplicity, we write "LLP wrt" and "RLP wrt" to mean "left lifting property with respect to" and "right lifting property with respect to" respectively.

Remark 1.73. In some textbook, Serre fibrations are defined to be maps having the RLP wrt all $i_0 : I^n \hookrightarrow I^n \times I$. The two definitions are equivalent. Note that $i_0 : I^n \hookrightarrow I^n \times I$ and $|\Lambda^n_k| \hookrightarrow |\Delta^n|$ are canonically homeomorphic to $i_0 : |\Delta^n| \hookrightarrow |\Delta^n| \times |\Delta^1|$ and $(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times |\Delta^1|) \hookrightarrow |\Delta^n| \times |\Delta^1|$ for $n \geq 0$ respectively.

A classical Theorem ?? about *anodyne extensions* tells us that the smallest saturated classes (see Definition 2.8) containing $(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times |\Delta^1|) \hookrightarrow |\Delta^n| \times |\Delta^1|$ and $(X \times 0) \cup (Y \times |\Delta^1|) \hookrightarrow X \times |\Delta^1|$ for any relative CW-complex $Y \subseteq X$ respectively are the same. We give a quick proof below as Lemma 1.74. Hence from the Definition 1.72, we see Serre fibrations have the RLP wrt all $i_0 : I^n \hookrightarrow I^n \times I$ where we choose $Y = \emptyset,\ X = |\Delta^n|$ and especially Serre fibrations are a special case of Hurewicz fibrations. However the converse is much more complicated in some sense (but below I will a simple proof). To prove the converse we need to assume some knowledge about simplicial sets.

Actually a map $p : X \to B$ has RLP wrt all $i_0 : I^n \hookrightarrow I^n \times I$ iff $\mathrm{Sing}_\bullet(p) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(B)$ is a *Kan complex*, for which it has the RLP wrt all $\Lambda^n_k \hookrightarrow \Delta^n$. Passing to the geometric realization, we see p will be a Serre fibration in the sense of Definition 1.72. Details can be found in the Kerodon page `https://kerodon.net/tag/021Q`.

However using Lemma 1.65 and Remark 1.66 we can give a simper proof. Assume p has the RLP wrt all $i_0 : I^n \to I^n \times I$ and we prove it has the RLP wrt all $(I^n \times 0) \cup (\partial I^n \times I) \hookrightarrow I^n \times I,\ n \geq 0$ which is canonically homeomorphic to $|\Lambda^n_k| \hookrightarrow |\Delta^n|,\ n \geq 1$. After proving this we see p is actually a Serre fibration in the sense of Definition 1.72. Since $(I^n \times 0) \cup (\partial I^n \times I) \hookrightarrow I^n \times I$ is a strong deformation retract, $\partial I^n \hookrightarrow I^n$ is a closed cofibration and then such map φ exists. Conditions of Lemma 1.65 are satisfied expect the fact that p is not a Hurewicz fibration. But that p has the RLP wrt all $i_0 : I^n \hookrightarrow I^n \times I$ is enough. Note that to apply Lemma 1.65, $A = (I^n \times 0) \cup (\partial I^n \times I) = \varphi^{-1}(0)$ and $X = I^n \times I$. Such lifting F will also exist and so does h.

Lemma 1.74.

$$A_2 := \big\{(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times |\Delta^1|) \hookrightarrow |\Delta^n| \times |\Delta^1|\big\}$$
$$A_3 := \big\{(X \times 0) \cup (Y \times |\Delta^1|) \hookrightarrow X \times |\Delta^1|\ ;\ Y \subseteq X \textit{ is a relative CW-complex.}\big\}$$

Let M_2 and M_3 denotes the smallest saturated classes of A_2 and A_3 respectively. Then $M_2 = M_3$.

Proof. Since $|\partial\Delta^n| \hookrightarrow |\Delta^n|$ is actually a relative CW-complex (see Definition 1.79), it's obvious to see $M_2 \subseteq M_3$. Hence it's enough to prove $M_3 \subseteq M_2$.

$|\Delta^1| \approx I$ is locally compact Hausdorff $\Rightarrow - \times |\Delta^1|$ commutes with arbitrary colimits.Then

$$\coprod \big((|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times |\Delta^1|)\big) = (\coprod |\Delta^n| \times 0) \cup (\coprod |\partial\Delta^n| \times |\Delta^1|)$$

and

$$\coprod(|\Delta^n| \times |\Delta^1|) = (\coprod |\Delta^n|) \times |\Delta^1|$$

Therefore

$$(\coprod |\Delta^n| \times 0) \cup (\coprod |\partial\Delta^n| \times |\Delta^1|) \hookrightarrow (\coprod |\Delta^n|) \times |\Delta^1|$$

belongs to M_2.

Suppose

$$\begin{array}{ccc} \coprod |\partial\Delta^n| & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \coprod |\Delta^n| & \longrightarrow & X \end{array}$$

is a pushout and then taking the functor $- \times |\Delta^1|$, we obtain a new pushout

$$\begin{array}{ccc} \coprod |\partial\Delta^n| \times |\Delta^1| & \longrightarrow & Y \times |\Delta^1| \\ \downarrow & & \downarrow \\ \coprod |\Delta^n| \times |\Delta^1| & \longrightarrow & X \times |\Delta^1| \end{array}$$

This implies the following diagram is a pushout

$$\begin{array}{ccc} (\coprod |\Delta^n| \times 0) \cup (\coprod |\partial\Delta^n| \times |\Delta^1|) & \longrightarrow & (X \times 0) \cup (Y \times |\Delta^1|) \\ \downarrow & & \downarrow \\ \coprod |\Delta^n| \times |\Delta^1| & \longrightarrow & X \times |\Delta^1| \end{array}$$

then $(X \times 0) \cup (Y \times |\Delta^1|) \hookrightarrow X \times |\Delta^1| \in M_2$.

For a general relative CW-complex $Y \subseteq X$, we have the pushout

$$\begin{array}{ccc} \coprod |\partial\Delta^n| & \longrightarrow & Y \cup sk_{n-1}X \\ \downarrow & & \downarrow \\ \coprod |\Delta^n| & \longrightarrow & Y \cup sk_n X \end{array}$$

and this implies the following diagram is a pushout

$$\begin{array}{ccc} (\coprod |\Delta^n| \times 0) \cup (\coprod |\partial\Delta^n| \times |\Delta^1|) & \longrightarrow & (X \times 0) \cup \big((Y \cup sk_{n-1}X) \times |\Delta^1|\big) \\ \downarrow & & \downarrow j_{n-1} \\ \coprod |\Delta^n| \times |\Delta^1| & \longrightarrow & (X \times 0) \cup \big((Y \cup sk_n X) \times |\Delta^1|\big) \end{array}$$

Taking the colimit of j_{n-1} we conclude $(X \times 0) \cup (Y \times |\Delta^1|) \hookrightarrow X \times |\Delta^1| \in M_2$. □

Before starting to prove Theorem 1.71, I want to state an important theorem for Serre fibrations which can help compute homotopy groups, though we will not need it in this section.

Theorem 1.75. *Assume $p : (X, x_0) \to (Y, y_0)$ is a Serre fibration and its fiber is $F = p^{-1}(y_0)$. Then for all $x \in F$ the following sequence is exact for groups when $n \geq 1$ and for pointed sets when $n = 0$, which is induced by $(F, x) \xrightarrow{i} (X, x) \xrightarrow{p} (Y, y_0)$.*

$$\dots \longrightarrow \pi_n(F, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{p_*} \pi_n(Y, y_0) \xrightarrow{\partial} \pi_{n-1}(F, x) \longrightarrow \dots \longrightarrow \pi_0(Y)$$

Proof. Suppose $[\alpha] \in \pi_n(X, x)$ such that $p_*([\alpha]) = [p \circ \alpha] = 0 \in \pi_n(Y, y_0)$. Then $p \circ \alpha \simeq \mathrm{cst}_{y_0}$ rel $|\partial\Delta^n|$. Then the lifting problem is solved.

$$\begin{array}{ccc} (|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times I) & \xrightarrow{(\alpha, \mathrm{cst}_x \circ pr_1)} & X \\ \downarrow & \overset{H}{\nearrow} & \downarrow p \\ |\Delta^n| \times I & \xrightarrow[p\circ\alpha \simeq \mathrm{cst}_{y_0} \text{ rel } |\partial\Delta^n|]{} & Y \end{array}$$

$\beta = H(-,1) \Rightarrow p \circ \beta = \mathrm{cst}_{y_0} \Rightarrow \mathrm{im}\ \beta \subseteq F$ and $H : \alpha \simeq \beta \ \mathrm{rel}\ |\partial\Delta^n|$ in X. This proves $\ker p_* \subseteq \mathrm{im}\ i_*$. Since $p \circ i = \mathrm{cst}_{y_0}$, $\mathrm{im}\ i_* \subseteq \ker p_*$ is trivial. Hence $\ker p_* = \mathrm{im}\ i_*$.

Next we should define $\partial : \pi_n(Y, y_0) \to \pi_{n-1}(F, x)$. For any $[\gamma] \in \pi_n(Y, y_0)$,

$$\begin{array}{ccc} |\Lambda^n_0| & \xrightarrow{\mathrm{cst}_x} & X \\ \big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ |\Delta^n| & \xrightarrow[\gamma]{} & Y \end{array}$$

we define $\partial[\gamma] = [\theta \circ d_0]$ where $d_0 : |\Delta^{n-1}| \hookrightarrow |\Delta^n|$ represents the 0-th face. At first we should check ∂ is well defined. Note that $\mathrm{im}\ \gamma|\ |\partial\Delta^n| = \{y_0\}$ and therefore $\theta \circ d_0 : (|\Delta^{n-1}|, |\partial\Delta^{n-1}|) \to (F, x)$. Next if $\gamma \simeq \gamma' \ \mathrm{rel}\ |\partial\Delta^n|$, the following diagram can be solved.

$$\begin{array}{ccc} (|\Delta^n| \times \{0,1\}) \cup (|\Lambda^n_0| \times I) & \xrightarrow{((\theta,\theta'),\mathrm{cst}_x)} & X \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ |\Delta^n| \times I & \xrightarrow[\gamma \simeq \gamma' \ \mathrm{rel}\ |\Delta^n|]{} & Y \end{array}$$

Note that $(|\Delta^n| \times \{0,1\}) \cup (|\Lambda^n_0| \times I)$ is canonically homeomorphic to $(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times I)$, since they are all obtained by digging out a face of the boundary of the cube I^{n+1}. Then $H \circ (d_0 \times \mathrm{id}_I) : \theta \circ d_0 \simeq \theta' \circ d_0 \ \mathrm{rel}\ |\partial\Delta^{n-1}|$, whichi lies in F.

In the following we will use Lemma 1.78 and we advise readers to read it first. $i_*\partial[\gamma] = [i \circ \theta d_0]$. But in X, θ is the lifting of $|\partial\Delta^n| \xrightarrow{(\theta d_0, x, \ldots, x)} X$ and from Lemma 1.78 we know θd_0 is trivial in $\pi_{n-1}(X, x)$. Then $i_* \circ \partial = 0$.

Conversely, if $[\alpha] \in \pi_{n-1}(F, x)$ satisfying $i_*[\alpha] = 0$, then by Lemma 1.78 we having the following diagram

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{(i\circ\alpha, x, \ldots, x)} & X \\ i\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ |\Delta^n| & \xrightarrow[p\circ\theta]{} & Y \end{array}$$

$p\theta d_0 = pi\alpha$. But $\mathrm{im}\ i \circ \alpha \subseteq p^{-1}(y_0)$. Then $p\theta d_0 = \mathrm{cst}_{y_0}$. Hence $[p \circ \theta] \in \pi_n(Y, y_0)$. From the definition of ∂ we see $\partial[p \circ \theta] = [\alpha]$. Then $\mathrm{im}\ \partial = \ker i_*$.

Assume $[\alpha] \in \pi_n(X, x)$, $\gamma = p \circ \alpha$ and then we see $\partial[p \circ \alpha] = [\alpha d_0]$ which is trivial. Therefore $\partial \circ p_* = 0$.

Conversely if $\partial[\gamma] = [\theta d_0] = 0$ in $\pi_{n-1}(F, x)$. Then the following diagrams are homotopic

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{(\theta d_0, x, \ldots, x)} & X \\ i\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ |\Delta^n| & \xrightarrow[\gamma]{} & Y \end{array} \quad \simeq \quad \begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ i\big\downarrow & & \big\downarrow p \\ |\Delta^n| & \xrightarrow[\gamma]{} & Y \end{array}$$

One of them has a lifting and then so is the other (see the proof of Proposition 1.77 for an explanation of this statement). Then for the second lifting problem, it has a solution $\mu : |\Delta^n| \to X$ satisfying $p \circ \mu = \gamma$. Then $p_*[\mu] = [\gamma]$. $\square$

Application 1.76. We use the long exact sequence induced by Serre fibrations to compute the homotopy groups of S^1. Consider the covering map $p : \mathbb{R} \to S^1,\ \theta \mapsto (\cos\theta, \sin\theta)$. We can suppose the base point of S^1 is $s_0 = (1,0)$ and then $F = p^{-1}(s_0) = \{2k\pi | k \in \mathbb{Z}\}$. Homotopy groups of F are trivial for $n \geq 1$ since S^n is connected. And note that $\mathbb{R}$ is *simply connected*. Hence in this case our long exact sequence is just

$$* \longrightarrow \pi_1(S^1, s_0) \xrightarrow{\partial} F \longrightarrow *$$

and we see for $n \geq 2$, $\pi_n(S^1, s_0) = 0$. Here $F \cong \mathbb{Z}$, but it is not equipped with group multiplication and we should prove ∂ is actually a group homorphism.

$$\begin{array}{ccc} * = |\Lambda^1_0| & \xrightarrow{\ 0\ } & \mathbb{R} \\ {\scriptstyle 0}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle p} \\ I = |\Delta^1| & \xrightarrow[\gamma]{} & S^1 \end{array}$$

$\theta \circ d_0$ is just $\theta(1)$. Since covering maps are Hurewicz fibrations with unique path lifting property, for $\gamma * \gamma'$ we can lift it pointwise which means we lift γ' first and then γ with constant map $\theta'(1) : * \to \mathbb{R}$. We conclude $\partial(\gamma * \gamma') = \gamma(1) + \gamma'(1)$. Then ∂ is actually a group homomorphism hence $\pi_1(S^1, s_0) \cong \mathbb{Z}$.

This induces a question when the fiber F can be equipped with a group structure. A limited situation is that for a covering map $p : E \to B$ if E is *simply connected* and *locally path connected*, then $\pi_1(B, b) \cong F \cong \mathrm{Aut}(p)$, whose proof uses the unique path lifting property deeply.

In the following we start to prove Theorem 1.71. At first we need to characterise trivial Serre fibrations.

Proposition 1.77. *$p : X \to Y$ is a trivial Serre fibration iff it has the RLP wrt all inclusions $|\partial\Delta^n| \hookrightarrow |\Delta^n|, n \geq 0, |\partial\Delta^0| = \emptyset$.*

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ |\Delta^n| & \longrightarrow & Y \end{array}$$

To prove this result we need the following lemma:

Lemma 1.78. *A map $\alpha : (|\Delta^n|, |\partial\Delta^n|) \to (X, x)$ represents the identity element (constant map) of $\pi_n(X, x)$ iff the following lifting problem can be solved.*

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha, x, \ldots, x)} & X \\ {\scriptstyle i}\big\downarrow & \overset{\beta}{\nearrow} & \\ |\Delta^{n+1}| & & \end{array}$$

where, $(\alpha, x, ..., x)$ describes $n+1$'s functions on faces of $|\Delta^{n+1}|$.

Proof of Lemma 1.78. "$\Rightarrow$". We simply denote the constant map $(Y, y) \to (X, x)$ by x if there is no confusion. Suppose $H : x \simeq \alpha \operatorname{rel} |\partial\Delta^n|$ and we want show the existence of the lifting β.

From Example 1.58 and Remark 1.73, for any spaces X, $X \to *$ is a Hurewicz fibration especially a Serre fibration.

$$\begin{array}{ccc} (|\Delta^{n+1}| \times 0) \cup (|\partial\Delta^{n+1}| \times I) & \xrightarrow{(x,(H,x,...,x))} & X \\ \big\downarrow & \overset{K}{\nearrow} & \\ |\Delta^{n+1}| \times I & & \end{array}$$

Let $\beta = K(-, 1)$ and then $\beta|\, |\partial\Delta^{n+1}| = (\alpha, x, ..., x)$.

"$\Leftarrow$". If such extension β exists,

$$\begin{array}{ccc} (|\Delta^{n+1}| \times \{0, 1\}) \cup (\left|\Lambda_0^{n+1}\right| \times I) & \xrightarrow{((x,\beta),x)} & X \\ \big\downarrow & \overset{K'}{\nearrow} & \\ |\Delta^{n+1}| \times I & & \end{array}$$

Then $K' \circ (d_0 \times \mathrm{id}_I) :\ x \simeq \alpha \operatorname{rel} |\partial\Delta^n|$. □

Proof of Proposition 1.77. "$\Rightarrow$". If $p : X \to Y$ is a trivial Serre fibration, then $p_* : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$. Given any point $y \in Y$, there is some $x \in X$ such that there exists some path connecting $p(x)$ and y. Since Serre fibrations has the path lifting property, the path has a lifting in X which means $p^{-1}(y)$ is not empty and p is surjective. This solves the problem when $n = 0$. In the following, we assume $n \geq 1$.

First, we prove a lemma that any two homotopic lifting problems are equivalent if p is a Serre fibration.

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{\alpha} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle p} \\ |\Delta^n| & \xrightarrow{\beta} & Y \end{array} \quad \simeq \quad \begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{\alpha'} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle p} \\ |\Delta^n| & \xrightarrow{\beta'} & Y \end{array}$$

They are homotopic if there is a homotopic diagram connecting them:

$$\begin{array}{ccc} |\partial\Delta^n| \times I & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ |\Delta^n| \times I & \longrightarrow & Y \end{array}$$

For two homotopic lifting problems, if one of them has a solution then so is the other. This is easy to see, since Serre fibrations has the RLP wrt all $(|\Delta^n| \times 0) \cup (|\partial\Delta^n| \times I) \hookrightarrow |\Delta^n| \times I$.

Given a lifting problem

$$\begin{CD} |\partial\Delta^n| @>{\alpha}>> X \\ @V{i}VV @VV{p}V \\ |\Delta^n| @>{\beta}>> Y \end{CD}$$

we start to deform α and find a new solvable lifting problem. $|\Lambda^n_k|$ is contractible which means it's homotopy equivalent to the one point space. Then we can find a homotopy $H : |\Lambda^n_0| \times I \to |\Lambda^n_0|$, $\mathrm{id}_{|\Lambda^n_0|} \simeq 0$ where 0 represents the constant map on the 0-th point $(1, 0, ..., 0)$ of $|\Delta^n|$. Hence

$$h_1 : |\Lambda^n_0| \times I \xrightarrow{H} |\Lambda^n_0| \hookrightarrow |\Delta^n| \xrightarrow{\alpha} X$$

is the homotopy of $\alpha|\ |\Lambda^n_0| \simeq \alpha(0) = \alpha(1, 0, ..., 0)$. According to Remark 1.73, since $|\Lambda^n_0| \hookrightarrow |\partial\Delta^n|$ is a relative CW-complex, the following lifting problem can be solved for the Serre fibration $X \to *$.

$$\begin{array}{ccc} (|\partial\Delta^n| \times \{0\}) \cup (|\Lambda^n_0| \times I)) & \xrightarrow{(\alpha, h_1)} & X \\ \big\downarrow & \nearrow_{h} & \\ |\partial\Delta^n| \times I & & \end{array}$$

Assume $\alpha_0 = h(-, 1) \circ d_0$, $x = \alpha(0)$ and then $h(-, 1) = (\alpha_0, x, .., x)$.

$$\begin{CD} |\partial\Delta^n| \times I @>{h}>> X \\ @VVV @VV{p}V \\ |\Delta^n| \times I @>{}>> Y \end{CD}$$

The dotted line is obtained by the extension of the strong deformation retract $(|\Delta^n| \times \{0\}) \cup (|\partial\Delta^n| \times I)$.

$$\begin{array}{ccc} (|\Delta^n| \times \{0\}) \cup |\partial\Delta^n| \times I) & \xrightarrow{(\beta, ph)} & Y \\ \big\downarrow & \nearrow_{K} & \\ |\Delta^n| \times I & & \end{array}$$

Let $\beta' = K(-, 1)$. Hence the original diagram is homotopic with

$$\begin{CD} |\partial\Delta^n| @>{(\alpha_0, x, ..., x)}>> X \\ @V{i}VV @VV{p}V \\ |\Delta^n| @>{\beta'}>> Y \end{CD}$$

Hence, by the Lemma 1.78, $[p \circ \alpha_0] = 0$ in $\pi_{n-1}(Y, p(x))$. For p is a weak equivalence, $[\alpha_0]$ is trivial in $\pi_{n-1}(X, x)$. Hence there is a homotopy $\alpha_0 \simeq x \text{ rel } |\partial\Delta^{n-1}|$, which will in fact give a homotopy $h' : (\alpha_0, x, ..., x) \simeq x$.

The diagram

$$\begin{array}{ccc} |\partial\Delta^n| \times I & \xrightarrow{h'} & X \\ \big\downarrow & & \big\downarrow p \\ |\Delta^n| \times I & \dashrightarrow & Y \end{array}$$

will give a new homotopic diagram

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \big\downarrow i & & \big\downarrow p \\ |\Delta^n| & \xrightarrow{\beta''} & Y \end{array}$$

That p is a weak equivalence implies p_* is surjective. Hence there is a map $\theta : (|\Delta^n|, |\partial\Delta^n|) \to (X, x)$ such that $[p \circ \theta] = [\beta'']$. This fact implies the diagram above is homotopic with the following diagram which has a lifting solution θ.

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \big\downarrow i & \overset{\theta}{\nearrow} & \big\downarrow p \\ |\Delta^n| & \xrightarrow{p\circ\theta} & Y \end{array}$$

Finally the original lifting problem is successfully solved.

"$\Leftarrow$". If $p : X \to Y$ has the RLP wrt all inclusions $|\partial\Delta^n| \hookrightarrow |\Delta^n|$, then it has the RLP wrt all relative CW-complexes $Y \subseteq X$ especially $|\Lambda^n_k| \hookrightarrow |\Delta^n|$. Hence it's a Serre fibration. Next we prove it's a weak equivalence.

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ i \big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ |\Delta^n| & \xrightarrow{\sigma} & Y \end{array}$$

The diagram above implies p_* is epic.

If $[\alpha] \in \pi_n(X, x)$, $[p \circ \alpha]$ is trivial in $\pi_n(Y, p(x))$, then by Lemma 1.78 there will be an extension:

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(p\alpha, p(x), \ldots, p(x))} & X \\ i \big\downarrow & \overset{\beta}{\nearrow} & \\ |\Delta^{n+1}| & & \end{array}$$

And we will have a lifting in the following diagram:

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha, x, \ldots, x)} & X \\ i \big\downarrow & \nearrow & \big\downarrow p \\ |\Delta^{n+1}| & \xrightarrow{\beta} & Y \end{array}$$

According to Lemma 1.78 again, $[\alpha]$ is trivial in $\pi_n(X,x)$. Hence, p_* is monic, then an isomorphism. □

Proof of Theorem 1.71. (M1). **Top** is complete and cocomplete due to the fact that products, coproducts, equalizers and coqualizers all exist in **Top**.

(M2). $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f = h$. Weak equivalences are homotopy isomorphisms. Hence, if f and g or g and h are weak equivalences, then h or f will also be a weak equivalence. We now assume f and h are weak equivalences. It's obvious to see $g_* : \pi_n(Y, f(x)) \to \pi_n(Z, gf(x))$ are isomorphisms. $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection between path components of the space X and Y. Then, for any $y \in Y$, there exists an $x \in X$ such that $f(x)$ and y are in the same path component and there is a path φ between them.

$$\begin{array}{ccc} \pi_n(Y, f(x)) & \xrightarrow{T_\varphi} & \pi_n(Y, y) \\ \big\downarrow g_* & & \big\downarrow g_* \\ \pi_n(Z, gf(x)) & \xrightarrow{T_{g\circ\varphi}} & \pi_n(Z, g(y)) \end{array}$$

Hence $g_* : \pi_n(Y,y) \to \pi_n(Z, g(y))$ is an isomorphism for any $y \in Y$, which means g is a weak equivalence.

(M3). We assume f is a retract of g.

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ \big\downarrow f & & \big\downarrow g & & \big\downarrow f \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

First, we let g be a weak equivalence.

$$\begin{array}{ccccc} \pi_n(X,x) & \xrightarrow{i_*} & \pi_n(Y,x) & \xrightarrow{r_*} & \pi_n(X,x) \\ \big\downarrow f_* & & \cong\big\downarrow g_* & & \big\downarrow f_* \\ \pi_n(X',x') & \xrightarrow{i'_*} & \pi_n(Y',x') & \xrightarrow{r'_*} & \pi_n(X',x') \end{array}$$

$r \circ i = \mathrm{id}_X$. Hence i_* is monic, which means $g_* \circ i_* = i'_* \circ f_*$ is monic. Therefore f_* is monic. The similar argument shows f_* is epic. Hence f_* is an isomorphism and f is a weak equivalence.

Since cofibrations and Serre fibrations are all defined by the lifting properties with respect to a certain class of morphisms, from Theorem 2.10 and Corollary 2.11 we conclude that they are closed under retraction.

To prove (M4), we need to prove (M5) first, but to prove (M5) we should state some properties of pushouts in preparation of the small object arguments.

Suppose the following diagram is a pushout in **Top**:

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ \big\downarrow i & & \big\downarrow j \\ Y & \xrightarrow{g} & B \end{array}$$

If i is monic, then so is j. In fact, $B = A \coprod Y / \sim$, where elements of $f^{-1}(a)$ are identified with a. Now, we want to prove the proposition that if $X \subseteq Y$ is a strong deformation retract, then $A \subseteq B$ will also be a strong deformation retract.

$H : Y \times I \to Y$, $i \circ r \simeq \mathrm{id}_Y \text{ rel } X$. We define $G : B \times I \to B$ by

$$G(b,t) = \begin{cases} g \circ H(y,t), & b = [y] \\ j(a) = [a], & b = [a] \end{cases}$$

The statement above is proved.

Now we start to prove every map $f : X \to Y$ can be factored as $f = p \circ i$ where p is a Serre fibration and i is a trivial cofibration. The proof of the other part of aixiom 5 is the same as this.

D_0 is the class consisting of all commutative diagrams of the following form:

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & X = X_0 \\ \Big\downarrow & & \Big\downarrow {\scriptstyle f = p_0} \\ |\Delta^n| & \longrightarrow & Y \end{array}$$

Then we will have the following pushout.

$$\begin{array}{ccc} \coprod_{D_0} |\Lambda^n_k| & \longrightarrow & X_0 \\ \Big\downarrow & & \Big\downarrow {\scriptstyle i_0} \\ \coprod_{D_0} |\Delta^n| & \longrightarrow & X_1 \\ & & \quad \searrow {\scriptstyle p_1} \\ & & \qquad\quad Y \end{array}$$

Clearly i_0 is monic. Next, we denote the set consisting of all commutative diagrams

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & X_1 \\ \Big\downarrow & & \Big\downarrow {\scriptstyle p_1} \\ |\Delta^n| & \longrightarrow & Y \end{array}$$

by D_1. Then, we will have the pushout X_2. Continuing this process, finally we will obtain the following commutative diagram:

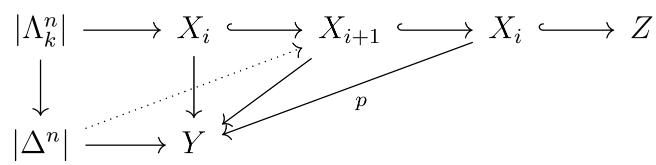
$$\begin{array}{ccccccc} X_0 & \xhookrightarrow{i_0} & X_1 & \xhookrightarrow{i_1} & \ldots & \hookrightarrow & \operatorname{colim} X_i = Z \\ \Big\downarrow {\scriptstyle p_0} & \swarrow {\scriptstyle p_1} & & & & \swarrow {\scriptstyle p} & \\ Y & & & & & & \end{array}$$

Now we want to prove $p : Z = \operatorname{colim} X_i \to Y$ is a Serre fibration. In fact, because $|\Lambda^n_k|$ is compact, we actually have natural isomorphisms, which are induced by the composition of morphisms, $\mathrm{colim}_i\ \mathrm{Hom}_{\mathbf{Top}}(|\Lambda^n_k|, X_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Top}}(|\Lambda^n_k|, \mathrm{colim}\ X_i)$. We prove that every morphism $|\Lambda^n_k| \to \mathrm{colim}\ X_i$ factors through some $|\Lambda^n_k| \to X_i$.[1]

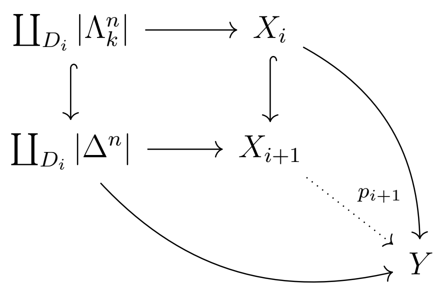
To be convenient, we may assume $X_0 = \emptyset$, or we can just deal with$X_{n+1} - X_n$. The two ways are equivalent. Given $|\Lambda^n_k| \to Z = \mathrm{colim}_i\ X_i$, the image in Z is compact and denoted by K. We will prove K just intersects finitely many disjoint simplexes $|\Delta^n|$.

If it's not ture, then there will be a subset S of K such that every element of S is in disjoint simplexes and it has infinitely many elements. We will prove S is closed i.e. $Z - S$ is open. It can be proved by induction. If $X_i - S$ is open, then due to the gluing process of pushouts, $X_{i+1} - S$ consists of the union of $X_i - S$ and the interior of some simplexes or simplexes minusing a point, which will also be open. Hence, S is compact. But, S is endowed with discrete topology and has infinitely many elements. It's impossible for S to be compact. Therefore, K just intersects finitely many disjoint simplexes $|\Delta^n|$, which means the morphism $|\Lambda^n_k| \to \mathrm{colim}\ X_i$ factors through some $|\Lambda^n_k| \to X_i$.

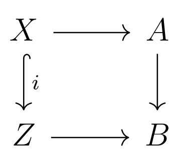
That $p : Z \to Y$ is a fibration is shown in the following diagram:



The dotted line is due to the follwing pushout.

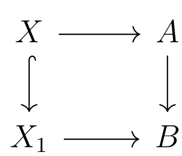


Next, we prove $i : X \hookrightarrow Z$ has the left lifting property with respect to all Serre fibrations. Given a lifting problem



we decompose it as a series of lifting problems.

First we solve the problem



[1]The proof comes from [Hov99a] Lemma 2.4.7.

Because $A \to B$ is a Serre fibration, there will be a lifting.

$$\begin{array}{ccccc} |\Lambda^n_k| & \longrightarrow & X & \longrightarrow & A \\ \big\downarrow & & \big\downarrow & \nearrow & \big\downarrow \\ |\Delta^n| & \longrightarrow & X_1 & \longrightarrow & B \end{array}$$

The coproduct of them will be the diagram:

$$\begin{array}{ccccc} \coprod_{D_0} |\Lambda^n_k| & \longrightarrow & X & \longrightarrow & A \\ \big\downarrow & & \big\downarrow & \nearrow & \big\downarrow \\ \coprod_{D_0} |\Delta^n| & \longrightarrow & X_1 & \longrightarrow & B \end{array}$$

Where X_1 is the pushout. Hence there will be a lifting $X_1 \to A$.

We solve all the lifting problems

$$\begin{array}{ccc} X_i & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ X_{i+1} & \longrightarrow & B \end{array}$$

and use the fact $Z = \operatorname{colim} X_i$ to solve the original lifting problem. Then it's done. Especially, $i \hookrightarrow Z$ is a cofibration.

Now, we only need to prove i is a weak equivalence. $|\partial\Delta^n|$ and $|\Delta^n|$ are compact, so any diagram

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & Z \\ \big\downarrow & \nearrow & \\ |\Delta^n| & & \end{array}$$

can be factored as

$$\begin{array}{ccccc} |\partial\Delta^n| & \longrightarrow & X_i & \longrightarrow & Z \\ \big\downarrow & \nearrow & & & \\ |\Delta^n| & & & & \end{array}$$

which means $\operatorname{colim}_i \pi_n(X_i, x) \cong \pi_n(Z, x)$.

Just like $X \hookrightarrow Z$, $X \hookrightarrow X_1$ also has the left lifting property with respect to all Serre fibrations. Therefore

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ \big\downarrow & \overset{r}{\nearrow} & \big\downarrow \\ X_1 & \longrightarrow & \{*\} \end{array}$$

X is a retract of X_1, which will be shown a strong deformation retract.[2]

We consider this pushout first.

$$\begin{array}{ccc} |\Lambda^n_k| & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow \big\uparrow \\ |\Delta^n| & \xrightarrow{g} & X' \end{array}$$

where $X' \subseteq X_1$.

To define the homotopy of strong deformation retract $H : X' \times I \to X'$ such that $H(x,t) = x$ if $x \in X$, we need to construct the function $h : |\Delta^n| \times I \to X'$ satisfying some conditions. We view $|\Delta^n|$ as a subset of X'. On $|\Lambda^n_k| \times I$ the function will be a projection forgetting I. On $|\Delta^n| \times \{1\}$ it will be the identity, and on $|\Delta^n| \times \{0\}$ it's $j \circ r$ where $j : X \hookrightarrow X'$, r is the retraction coming from $X_1 \to X$.

$$\begin{array}{ccc} (|\Delta^n| \times \{0,1\}) \cup (|\Lambda^n_k| \times I) & \longrightarrow & X' \\ \big\downarrow & \overset{h}{\nearrow} & \\ |\Delta^n| \times I & & \end{array}$$

Let $H||\Delta^n| \times I = h$, which gives the desired homotopy $j \circ r \simeq \mathrm{id}_{X'} \operatorname{rel} X$. Hence, $\pi_n(X,x) \cong \pi_n(X',x) \cong ... \cong \pi_n(X_1,x) \cong ... \cong \pi_n(Z,x)$. Then, $i : X \hookrightarrow Z$ is a weak equivalence. (M5) is proved.

(M4). We only need to prove the following lifting problem can be solved:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow i & & \big\downarrow p \\ B & \longrightarrow & Y \end{array}$$

where i is a trivial cofibration and p is a fibration.

Decompose i as

$$A \xrightarrow{j} E \xrightarrow{q} B$$

where j is a trivial cofibration and q is a fibration. $q \circ j = i$ is a weak equivalence. Hence, q is a trivial fibration. Then the lifting problem can be solved as

$$\begin{array}{ccccc} & & A & \longrightarrow & X \\ & & \big\downarrow j & \overset{v}{\nearrow} & \big| \\ A & \xrightarrow{j} & E & & \big| p \\ \big\downarrow i & \overset{u}{\nearrow} & \big\downarrow q & \searrow & \big\downarrow \\ B & \xrightarrow{\mathrm{id}} & B & \longrightarrow & Y \end{array}$$

[2]The idea to use the strong deformation retract is inspired by [Qui67] Chapter II, P3.4, Lemma 4.

The original lifting problem can be solved.

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{v\circ u}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

$\square$

Finally in this section, we talk about some basic facts about CW-complexes.[3]

Definition 1.79. *Given an inclusion $A \hookrightarrow X$, (A, X) is called a **relative CW-complex** if:*

1. *There is a decomposition of the space X, such that $A = X^{-1} \subseteq X^0 \subseteq ... \subseteq X$*
2. *$X = \operatorname{colim}_{i \geq -1} X^i$ and X has the induced topology, which means $U \subseteq X$ is open iff $U \cap X^i$ is open for all $i \geq -1$.*
3. *For all $n \geq 0$, the following diagram is a pushout:*

$$\begin{array}{ccc} \coprod_{j\in J} |\partial\Delta^n| & \longrightarrow & X^{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{j\in J} |\Delta^n| & \longrightarrow & X^n \end{array}$$

From axiom 3, we can conclude $X^n - X^{n-1} \approx \coprod_{j\in J}(|\Delta^n| - |\partial\Delta^n|)$. Therefore we call the process of axiom 3 gluing cells. Due to the property of colimits and pushouts, the topology of X is determined by the inclusion $A \hookrightarrow X$ and all maps $|\Delta^n| \to X$. A subset $U \subseteq X$ is open iff $U \cap A$ and all $U \cap |\Delta^n|$ are open.

Proposition 1.80. *The inclusion $A \hookrightarrow X$ of a relative CW-complex is a cofibration.*

Proof. $p : Y \to Z$ is a trivial fibration. According to Proposition 1.77, p has the RLP wrt all $|\partial\Delta^n| \hookrightarrow |\Delta^n|$, which means $|\partial\Delta^n| \hookrightarrow |\Delta^n|$ is a cofibration. Hence $\coprod |\partial\Delta^n| \hookrightarrow \coprod |\Delta^n|$ will also be a cofibration. Cofibrations are stable under pushouts[4]. Then $X^{n-1} \hookrightarrow X^n$ is a cofibration. Any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \big\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & Z \end{array}$$

can be solved by the universal property of colimits. This implies $A \hookrightarrow X$ is a cofibration. $\square$

Definition 1.81. *A CW-complex is a relative CW-complex $(\emptyset, X)$.*

Corollary 1.82. *In* **Top**, *any CW-complex is cofibrant.*

[3]See [Hat02] Appendix for more details.
[4]Use the universal property of pushouts.

Theorem 1.83. *For any relative CW-complex (A, X), if A is Hausdorff or normal then X will also be Hausdorff or normal.*

Sketch of the Proof. See [Hat02] P522 Proposition A.3 for the process of construction.

The proof of Hausdorff property and normal property are the same. Hence, we may assume A is normal. Then given any disjoint closed subsets K_1, K_2 of X, $K_i \cap A$ are closed. Hence, we can find two disjoint open subsets of A to separate $K_i \cap A$. If we have found such open subsets of X^{n-1}, then we should try to find disjoint open subsets of X^n to separate $K_i \cap X^n$. This can be done by considering the preimage of such open subsets in X^n. Consider the pushout diagram in Definition 1.79, we should construct two good enough disjoint open sets in $|\Delta^n|$, where any one of the open subsets consists of two parts. The one part is contained in $|\Delta^n| - |\partial\Delta^n|$, and the other contains the preimage of open subsets of X^{n-1} in $\partial\Delta^n$. Finally, the countable union of open subsets in each X^n is just what we want. □

Corollary 1.84. *Every CW-complex is normal, hence Hausdorff.*

2 Homotopical Algebra

In this chapter we talk about the abstract theory of homotopical algebra which is also called the theory of model categories and all of contents here are already in [Qui67].

Roughly speaking, in a model category there are three important classes of morphisms, called fibrations, cofibrations and weak equivalences respectively. They reveal the lifting properties and quasi-isomorphisms in a given category. For lifting property, we have seen its importance in previous sections. But in fact here the most important class of morphisms is that of weak equivalences. In general fibrations and cofibrations are defined to help us study properties of weak equivalences, which is just similar to that to study manifolds coordinates are not a must but they can really help us study manifolds. For a given manifold, there are many choices of local coordinates and for a class of weak equivalences there may also be some choices of fibrations and cofibrations to make them form a model category, which means in general for a category the model category structure is not unique. In this section we will give the example of the category of chain complexes and there are projective and injective model category structures on it.

Homotopical algebra is to study weak equivalences and some properties invariant under weak equivalences. Hence it's natural to look at localization categories with respect to weak equivalences first.

Definition 2.1. *Let $\mathcal{C}$ be a category with small* Hom *sets, and $\mathcal{W}$ be a set of morphisms. Then there will exist* ***the category of fractions*** *(or called* ***localization category****) $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ with respect to $\mathcal{W}$ and a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that:*

(1) For any $f \in \mathcal{W}$, $\gamma(f)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.

(2) For any functor $G : \mathcal{C} \to \mathcal{D}$ such that $\forall f \in \mathcal{W}, G(f)$ are isomorphims, there is a unique functor $F : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $F \circ \gamma = G$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \mathcal{C}[\mathcal{W}^{-1}] \\ {\scriptstyle G}\downarrow & \swarrow {\scriptstyle \exists! \, F} & \\ \mathcal{D} & & \end{array}$$

Remark 2.2. Obviously we know adding isomorphisms to $\mathcal{W}$ will not affect the universal category $\mathcal{C}[\mathcal{W}^{-1}]$. And we can enlarge $\mathcal{W}$ to become a subcategory of $\mathcal{C}$ and this will not affect the localization category as well. Moreover such localization category always exists though there will be some set theoretic difficulties. We give the proof of existence here.

Proof. We construct the category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ as follows. We consider a directed graph $\mathcal{G}$ first. The vertexes in $\mathcal{G}$ are just objects in $\mathcal{C}$. For any map $f : A \to B$ in $\mathcal{W}$, we add a map $f^{-1} : B \to A$. The set of oriented edges in $\mathcal{G}$ consists of edges in $\mathcal{C}$ and those f^{-1}. Then, we identify the path $f \circ f^{-1}$ with id_B , $f^{-1} \circ f$ with id_A , $g \circ h$ with gh where $g, h \in \mathrm{Mor}(\mathcal{C})$, $\mathrm{id} \circ g$ with g, $h \circ \mathrm{id}$ with h where $g, h \in \mathcal{G}$ and $g \circ (h \circ l)$ with $(g \circ h) \circ l$ for any oriented edges $g, h, l \in \mathcal{G}$. The quotient directed graph will be the category $\mathcal{C}[\mathcal{W}^{-1}]$.

There is another description of the Hom set of localization categories. In $\mathcal{C}[\mathcal{W}^{-1}]$ every morphism $X \to Y$ has the following form:

$$X \longleftarrow X_1 \longrightarrow X_2 \longleftarrow X_3 \longrightarrow \dots \longleftarrow X_n \longrightarrow Y$$

where left arrows are in $\mathcal{W}$, right arrows in $\mathrm{Mor}(\mathcal{C})$. Sequences obtained by adding identities are viewed the same as the original one. Hence for any two sequences we can add identities to them to make them having the same number of objects. So that we can define a complicated equivalence relation among such sequences. This equivalence relation is generated by the following diagram

$$\begin{array}{ccccccccc} & & X_1 & \longrightarrow & X_2 & \longleftarrow \dots \longrightarrow & X_n & & \\ & \swarrow & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & \searrow & \\ X & & & & & & & & Y \\ & \nwarrow & & & & & & \nearrow & \\ & & X_1' & \longrightarrow & X_2' & \longleftarrow \dots \longrightarrow & X_n' & & \end{array}$$

where vertical morphisms are in $\mathcal{W}$. Two sequences are relevant if there is such a commutative diagram above in $\mathcal{C}$. The equivalence is generated by these relations. □

Remark 2.3. The Definition 2.1 is strict since all such $\mathcal{C}[\mathcal{W}^{-1}]$'s are isomorphic which means there will exist one-to-one relations on objects and morphisms. But for categories we only consider equivalence classes of them not isomorphism classes. Therefore we give a weaker definition here.

Let $\mathrm{Hom}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ be the full subcategory of the category of functors between $\mathcal{C}$ and $\mathcal{D}$ consisting of those functors taking every morphism in $\mathcal{W}$ to isomorphisms. Then $\gamma : \mathcal{C} \to \mathbf{C}[\mathcal{W}^{-1}]$ is defined to have the universal property

$$\gamma^* : \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}}(\mathcal{C}, \mathcal{D}) \tag{5}$$

where γ^* is defined by composition and it's an equivalence between categories not an isomorphism. This will define $\mathcal{C}[\mathcal{W}^{-1}]$ up to equivalence.

Example 2.4. The category of all small categories is denoted by $\mathbf{Cat}$ and The set of equivalences between categories is denoted by $\mathcal{W}$. For any two functor $F, G : \mathcal{C} \to \mathcal{D}$ we say they are equivalent if there is a natural isomorphism $\tau : F \overset{\sim}{\Rightarrow} G$. It's obvious to see it's actually an equivalence relation among $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ and it's preserved by compositions. Hence we can define the homotopy category $\mathrm{Ho}(\mathbf{Cat})$ to have the same objects as $\mathbf{Cat}$ and its morphism sets are the equivalence classes described above. Then $\mathrm{Ho}(\mathbf{Cat}) \cong \mathbf{Cat}[\mathcal{W}^{-1}]$.

Proof. The category generated by one isomorphism $1 \xrightarrow{\sim} 2$ is denoted by $\bar{I}$. We write the functor category of $\mathcal{D}$ over $\bar{I}$ as $\mathcal{D}^{\bar{I}}$. Then the proof is similar to Theorem 1.3 and Theorem 3.20. It's enough to prove for any functor $\mathcal{F} : \mathbf{Cat} \to \mathcal{E}$ taking categorical equivalences to isomorphisms, if there is a natural isomorphism between functors $\tau : G \overset{\sim}{\Rightarrow} H$ where $G, H \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$, then $\mathcal{F}(G) = \mathcal{F}(H)$.

$$\begin{array}{ccccc} & & \mathcal{D}^{\bar{I}} & & \\ & \overset{\tau}{\nearrow} & \downarrow p & \overset{s}{\underset{\sim}{\nwarrow}} & \\ \mathcal{C} & \xrightarrow[(G,H)]{} & \mathcal{D} \times \mathcal{D} & \xleftarrow[\Delta]{} & \mathcal{D} \\ & & \downarrow\downarrow & & \\ & & \mathcal{D} & & \end{array}$$

where $\tau : a \mapsto (\tau_a : G(a) \xrightarrow{\sim} H(a))$, $p : (x \xrightarrow{\sim} y) \mapsto (x, y)$, $\Delta(x) = (x, x)$ and $s : x \to (\mathrm{id}_x : x \xrightarrow{\sim} x)$. Note s is actually a categorical equivalence. Obviously it's fully faithful. For any $y \xrightarrow{\sim} y'$ in $\mathcal{D}^{\bar{I}}$

$$\begin{array}{ccc} y & =\!=\!= & y \\ \| & & \downarrow \wr \\ y & \xrightarrow{\sim} & y' \end{array}$$

We see s is also essentially surjective hence a categorical equivalence. Then $\mathcal{F}(s)$ is an isomorphism. We are done. □

Remark 2.5. Note that localization of categories often cause set theoretic problems, which means the morphism set of $\mathcal{C}[\mathcal{W}^{-1}]$ may be a proper class. Example 2.4 is a special case since the morphism set of $\mathrm{Ho}(\mathbf{Cat})$ is small. Actually $\mathbf{Cat}$ is a model category which we will talked about later and the localization of model categories with respect to weak equivalences will not cause this set theoretic problem. What's more apart from the technique of model categories, there is a useful technique to solve this set theoretic problem as well which is actually earlier than Quillen's work and whose motivation is more natural. Such technique is called *calculus of fractions*. You can find details in Section 3.2.

Next to talk about model categories, let us begin with the *factorization system* first.

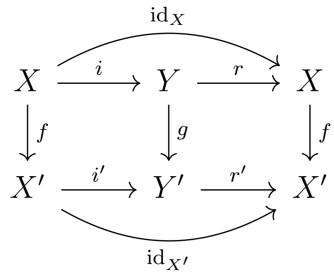
2.1 Factorization Systems

In a category $\mathcal{C}$, let $i : A \to B$ and $p : X \to Y$ be two morphisms in it. We say i has the *left lifting property* with respect to (LLP wrt) p or p has the *right lifting property* with respect to (RLP wrt) i if for any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

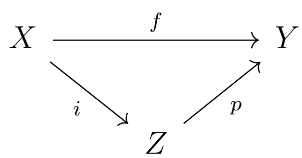
there exists $h : B \to X$ making the new diagram commutative. If $\mathcal{F}$ is a class of morphisms in $\mathcal{C}$, we use $l(\mathcal{F})$ (resp. $r(\mathcal{F})$) to denotes the class of morphisms having the LLP (resp. RLP) wrt all morphisms in $\mathcal{F}$. [5]

Definition 2.6. *In a category $\mathcal{C}$, we say $f : X \to X'$ is a* ***retract*** *of $g : Y \to Y'$ if there is the following comutative diagram:*

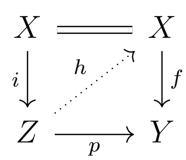


such that $r \circ i = \mathrm{id}_X, r' \circ i' = \mathrm{id}_{X'}$.

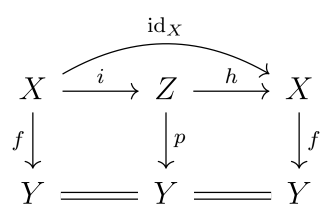
Lemma 2.7. *In a category $\mathcal{C}$, if $f : X \to Y$ can be factored as $f = p \circ i$ where f has the RLP (resp. LLP) wrt i (resp. p), then f is a retract of p (resp. i).*



Proof. We only need to assume $f \in r(i)$, since on the other hand we can deal with this problem in $\mathcal{C}^{op}$.



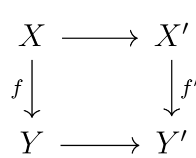
This diagram above implies



□

Now we want to talk about properties of $l(\mathcal{F})$.

Definition 2.8. *A class of morphisms $\mathcal{F}$ is closed under pushouts if given any pushout diagram,*



[5] In different textbooks, notations of $l(\mathcal{F})$ and $r(\mathcal{F})$ are different (see [Cis19], [Hov99a] and [Rie14]). Here we follow [Cis19] since we think his notations are the simplest.

that $f \in \mathcal{F}$ implies $f' \in \mathcal{F}$; it's closed under retraction if in the diagram of Definition 2.6, that $g \in \mathcal{F}$ implies $f \in \mathcal{F}$; it's closed under coproducts if given $f_i : X_i \to Y_i$ belonging to $\mathcal{F}$ for $i \in I$, then so does

$$\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

*$\mathcal{F}$ is closed under **transfinite compositions** if for every well-ordered set I with the initial element 0, for any functor $X : I \to \mathcal{C}$ such that for any element $i \in I, i \neq 0$, the colimit $\operatorname{colim}_{j<i} X(j)$ exists and the induced map*

$$\operatorname{colim}_{j<i} X(j) \to X(i)$$

is in $\mathcal{F}$, then the colimit $\operatorname{colim}_{i\in I} X(i)$ exists and the morphism $X(0) \to \operatorname{colim}_{i\in I} X(i)$ belongs to $\mathcal{F}$.

*The class of morphisms satisfying properties above is called **saturated**.*

Remark 2.9. Actually given a class of morphisms $\mathcal{F}$, if it's closed under pushouts and transfinite compositions, then it will also be closed under coproducts.

Proof. Suppose there are morphisms $f_i : X_i \to Y_i$ belonging to $\mathcal{F}$ for $i \in I$. From the well-ordering axiom, we may assume I is well-ordered and 0 is its initial element. Firstly, we have a pushout diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & \coprod_{i\in I} X_i \\ {\scriptstyle f_0}\downarrow & & \downarrow \\ Y_0 & \longrightarrow & Z_0 \end{array}$$

where Z_0 is actually $Y_0 \coprod_{i\in I\setminus 0} X_i$. If 1 is the successor of 0, then we have the pushout

$$\begin{array}{ccc} X_1 & \longrightarrow & Z_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow \\ Y_1 & \longrightarrow & Z_1 \end{array}$$

where $X_1 \to Z_0$ is just $X_1 \to \coprod_{i\in I} X_i \to Z_0$. For a limit number i, we let Z_i be the following poushout

$$\begin{array}{ccc} X_i & \longrightarrow & \operatorname{colim}_{j<i} Z_j \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ Y_i & \longrightarrow & Z_i \end{array}$$

Since $\mathcal{F}$ is closed under pushouts, $\coprod_{i\in I} X_i \to Z_0$, $Z_0 \to Z_1$ and $\operatorname{colim}_{j<i} Z_j \to Z_i$ all belong to $\mathcal{F}$. Note that $\operatorname{colim}_{j<i} Z_j$ is actually $(\coprod_{j<i} Y_j) \coprod (\coprod_{i'\geq i} X_{i'})$. Finally we see $\operatorname{colim}_{i\in I} Z_i = \coprod_{i\in I} Y_i$, which

can also be proved by the universal property of coproducts of Y_i's. Then that $\mathcal{F}$ is closed under transfinite compositions implies $\coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is in $\mathcal{F}$. □

Theorem 2.10. *In a category $\mathcal{C}$, for any class of morphisms $\mathcal{F}$, $l(\mathcal{F})$ is saturated.*

Proof. From Remark 2.9 we only to check $l(\mathcal{F})$ is closed under retraction, pushouts and transfinite compositions.

Step 1 (retraction). If f is the retraction of g where $g \in l\mathcal{F}$, given any $p : A \to B$ belonging to $\mathcal{F}$

$$\begin{array}{ccccccc} & & & \overset{\mathrm{id}_X}{\frown} & & & \\ X & \xrightarrow{i} & Y & \xrightarrow{r} & X & \xrightarrow{\alpha} & A \\ \downarrow f & & \downarrow g & \overset{\theta}{\nearrow} & \downarrow f & & \downarrow p \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' & \xrightarrow{\beta} & B \\ & & & \underset{\mathrm{id}_{X'}}{\smile} & & & \end{array}$$

Then there will be a lifting $\theta : Y' \to A$ and $\theta \circ i'$ will give the solution of the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ \downarrow f & & \downarrow p \\ X' & \xrightarrow{\beta} & B \end{array}$$

Hence $f \in l(\mathcal{F})$.

Step 2 (pushouts). Look at the pushout diagram in Definition 2.8 where $f \in l(\mathcal{F})$ and f' is the pushout of f. Also given the lifting problem above

$$\begin{array}{ccccc} X' & \longrightarrow & X & \xrightarrow{\alpha} & A \\ f' \downarrow & \overset{\theta}{\nearrow} & \downarrow f & \overset{\mu}{\nearrow} & \downarrow p \\ Y' & \longrightarrow & Y & \xrightarrow[\beta]{} & B \end{array}$$

θ is induced by f. Using the universal property of pushouts, we see the solution μ exists.

Step 3 (transfinite compositions). If 1 is the successor of 0,

$$\begin{array}{ccc} X_0 & \longrightarrow & A \\ \downarrow & \nearrow & \\ X_1 & & \downarrow p \\ \downarrow & \searrow & \\ \vdots & & \\ \downarrow & & \\ \operatorname{colim}_{i\in I} X_i & \longrightarrow & B \end{array}$$

the lifting respect to $X_0 \to X_1$ exists. Especially for any $X_i \to X_{i+1}$ where $i+1$ is the the successor of i, the lifting $X_{i+1} \to A$ exists. If i is a limit number, then from the univsersal property of colimits we see there will exist $\operatorname{colim}_{j<i} X_j \to A$. And then since $\operatorname{colim}_{j<i} X_j \to X_i$ belongs to $l(\mathcal{F})$, the lifting $X_i \to A$ exists. Finally there will be some $\operatorname{colim}_{i \in I} X_i \to A$ making the diagram commutative. $\square$

Corollary 2.11. *In a category $\mathcal{C}$, for any class of morphisms $\mathcal{F}$, $r(\mathcal{F})$ is closed under retraction, pullbacks, products and the dual process of transfinite compositions especially finite compositions.*

Proof. Apply Theorem 2.10 to $\mathcal{C}^{op}$. $\square$

Fact 2.12. *In a category $\mathcal{C}$ there are two classes of morphisms $\mathcal{F}$ and $\mathcal{F}'$, then*

(1) $\mathcal{F} \subseteq r(\mathcal{F}') \Leftrightarrow \mathcal{F}' \subseteq l(\mathcal{F})$

(2) $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow l(\mathcal{F}') \subseteq l\mathcal{F}$.

(3) $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow r(\mathcal{F}') \subseteq r\mathcal{F}$.

(4) $r(\mathcal{F}) = r \circ l \circ r(\mathcal{F})$.

(5) $l(\mathcal{F}) = l \circ r \circ l(\mathcal{F})$.

Proof. We only prove the property of (4). Since all morphisms in $\mathcal{F}$ have the LLP wrt $r(\mathcal{F})$, then $\mathcal{F} \subseteq l \circ r(\mathcal{F})$. This implies $r \circ l \circ r(\mathcal{F}) \subseteq r(\mathcal{F})$. Replacing $\mathcal{F}$ by $r(\mathcal{F})$, we see it's clear that $r(\mathcal{F}) \subseteq r \circ l(r(\mathcal{F}))$. $\square$

Definition 2.13. *A **weak factorization system** in a category $\mathcal{C}$ is a couple $(\mathcal{F}, \mathcal{G})$ of classes of morphisms satisfying*

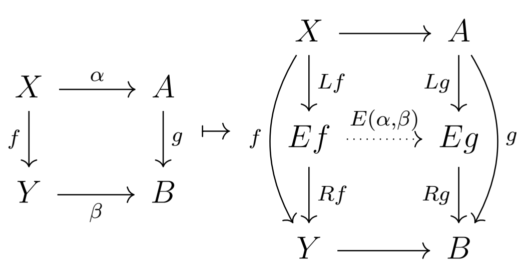
(1) both $\mathcal{F}$ and $\mathcal{G}$ are closed under retraction.

(2) $\mathcal{F} \subseteq l(\mathcal{G}) (\Leftrightarrow \mathcal{G} \subseteq r(\mathcal{F})$.

(3) any morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a factorization $f = p \circ i$ where $i \in \mathcal{F}$ and $p \in \mathcal{G}$.

In the most cases, we may require the factorization $f = p \circ i$ to be functorial and we will explain what's the meaning of *functorial factorization system*. In the factorization $f = p \circ i$, we use Rf and Lf to denote p and i respectively.

Definition 2.14. *In a category $\mathcal{C}$, a **functorial factorization system** is a weak factorization system with a functor $\mathcal{C}^2 \to \mathcal{C}^3$ from the category of arrows in $\mathcal{C}$ to the category of composable pairs of arrows in $\mathcal{C}$.*



This definition actually means the replacement functors L and R are functorial. The reason why we often require this property is practical. The method Quillen uses to construct a model category is called *small object arguments*. From this method, we always obtain a functorial factorization system, since every step of this method is functorial. Now let us introduce this method which can also be found in [Cis19] Proposition 2.1.9, [Hov99a] Theorem 2.1.14 and [Rie14] Theorem 12.2.2.

Definition 2.15. *Given a cardinal κ, a non-empty partially ordered set E is κ-**filtered** if for any family of its elements x_j indexed by J with $|J| < \kappa$, then there exists an element $x \in E$ such that $x_j \leq x$ for all $j \in J$.*

Theorem 2.16 (Small Object Argument)**.** *Let $\mathcal{C}$ be a locally small category with small colimits, equipped with a small set of morphisms $\mathcal{F}$. If there exists a cardinal κ such that for any element $i : K \to L$ in $\mathcal{F}$, the functor*

$$\mathrm{Hom}_{\mathcal{C}}(K, -) : \mathcal{C} \to \mathbf{Sets}$$

commutes with colimits indexed by κ-filtered well-ordered sets, then the couple $(l \circ r(\mathcal{F}), r(\mathcal{F}))$ forms a functorial factorization system and $l \circ r(\mathcal{F})$ is the smallest saturated class containing $\mathcal{F}$.

Proof. Suppose κ exists and $\lambda \geq \kappa$. Given any morphism $f : X \to Y$ in $\mathrm{Mor}(\mathcal{C})$, let D_0 be the class consisting of all commutative diagrams

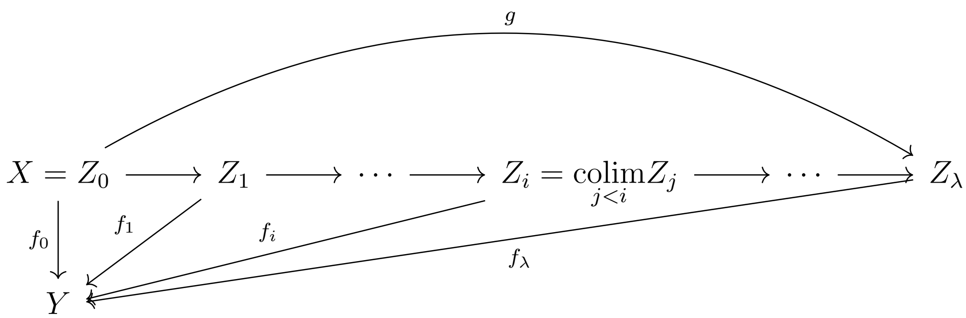
$$\begin{array}{ccc} K & \longrightarrow & X = Z_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f = f_0} \\ L & \longrightarrow & Y \end{array}$$

with $i \in \mathcal{F}$. Then let Z_1 be the pushout of

$$\begin{array}{ccccc} \coprod_{D_0} K & \longrightarrow & X = Z_0 & & \\ \downarrow & & \downarrow & \searrow{\scriptstyle f_0} & \\ \coprod_{D_0} L & \longrightarrow & Z_1 & \overset{\exists !}{\underset{f_1}{\dashrightarrow}} & Y \\ & \searrow & & \nearrow & \end{array}$$

If $i + 1$ is the successor of i, using this method to obtain Z_{i+1} and $f_{i+1} : Z_{i+1} \to Y$ from Z_i and f_i. If i is a limit number, define $Z_i = \operatorname*{colim}_{j<i} Z_j$ and this will induce a morphism

$f_i : Z_i \to Y$ from f_j's. Finally we have the following factorization

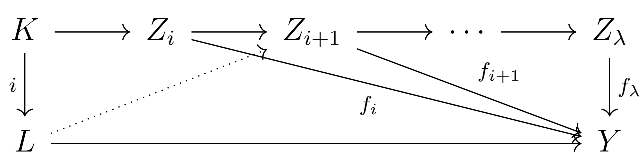


We prove $g \in l \circ r(\mathcal{F})$ and $f_\lambda \in r(\mathcal{F})$.

Given a lifting problem

$$\begin{array}{ccc} K & \longrightarrow & Z_\lambda \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ L & \longrightarrow & Y \end{array}$$

with $i \in \mathcal{F}$. Since $\mathrm{Hom}_{\mathcal{C}}(K, -)$ commutes with λ-colimits, $K \to Z_\lambda = \underset{j<\lambda}{\mathrm{colim}} Z_j$ factors through some Z_i.



But from the definition of Z_{i+1}, we see $i : K \to L$ here belongs to D_i, which means there exists a lifting $L \to Z_{i+1}$ whose composition with $Z_{i+1} \to Z_\lambda$ gives the lifting $L \to Z_\lambda$.

Next we should prove $g \in l \circ r(\mathcal{F})$. It's obvious to see $\mathcal{F} \subseteq l \circ r(\mathcal{F})$. From Theorem 2.10,$l \circ r(\mathcal{F})$ is saturated. But $g : X \to Z_\lambda$ is obtained by "attaching cells" of $\coprod_{D_i} K \to \coprod_{D_i} L$. Therefore it's clear $g \in l \circ r(\mathcal{F})$.

The statements in the previous paragraph obviously imply that the smallest saturated class of $\mathcal{F}$ is contained in $l \circ r(\mathcal{F})$. On the other hand, assume $f : X \to Y$ belonging to $l \circ r(\mathcal{F})$, then $f = p \circ i$ where $i \in l \circ r(\mathcal{F})$ and $p \circ r(\mathcal{F})$ which we haved proved above. Note that from the proof, i is in the smallest saturated class of $\mathcal{F}$. Since f has the LLP wrt p, from Lemma 2.7 f is a retract of i hence belonging to the smallest saturated class of $\mathcal{F}$ by Definition 2.8.

Finally we should prove this factorization talked above is functorial. Given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

then we have classed D_0 and D'_0 for f and f' respectively. But it's clear there is a map $D_0 \to D'_0$ via compositions and this will induce a map $Z_1 \to Z'_1$. And finally we will obtain $Z_\lambda \to Z'_\lambda$. Since in every step of this process the morphism is induced by the universal property, it's clear the final map $Z_\lambda \to Z'_\lambda$ is functorial. □

Application 2.17. When proving **Top** has Quillen's model category structure in Theorem 1.71, $\kappa = \aleph_0$ since domains of morphisms in $\mathcal{F}$ there are compact spaces.

Now let us introduce the concept of model categories.

Definition 2.18. *A* ***model category*** *$\mathcal{M}$ has three classes of morphisms which are denoted by Cof, Fib and $\mathcal{W}$, and are called cofibtrations, fibrations and weak equivalences respectively. Moreover it satisfies the following axioms*

(M1) $\mathcal{M}$ has all finite limits and colimits.

(M2) In the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{h} & \downarrow g \\ & & Z \end{array}$$

If any two of the three morphisms f, g and $h = g \circ f$ are weak equivalences, then so is the other. This property is called "two out of three".

(M3) Cof, Fib and $\mathcal{W}$ are closed under retraction.

(M4) $Cof \subseteq l(Fib \cap \mathcal{W})$ and $Cof \cap \mathcal{W} \subseteq l(Fib)$.

(M5) Every morphism $f : X \to Y$ in $\mathcal{M}$ can be factored as $f = p \circ i = q \circ j$ such that $i \in Cof \cap \mathcal{W}$, $p \in Fib$, $j \in Cof$ and $q \in Fib \cap \mathcal{W}$.

Note that axioms (M4) and (M5) actually mean $(Cof, Fib \cap \mathcal{W})$ and $(Cof \cap \mathcal{W}, Fib)$ form two weak factorization systems. Morphisms in $Cof \cap \mathcal{W}$ (resp. $Fib \cap \mathcal{W}$) are called *trivial cofibrations* (resp. *trivial fibrations*). In a model category we use $\emptyset$ and $*$ to denote its initial object and terminal object respectively. An object $X \in \mathcal{M}$ is **cofibrant** (resp. *fibrant*) if $\emptyset \to X$ (resp. $X \to *$) is a cofibration (resp. fibration).

2.2 Model Category Structure on $\mathbf{Ch}_{\geq 0}(R)$

In this section we will use Theorem 2.16 to prove $\mathbf{Ch}_{\geq 0}(R)$ is a model category.

$\mathbf{Ch}_{\geq \mathbf{0}}(R)$ denotes the full subcategory of $\mathbf{Ch}(R)$ consisting of all complexes of left R-modules such that $C_n = 0$ for $n < 0$. In this section, we will sketch the proof that $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ is a model category, which is much easier then the proof on **Top**. Actually $\mathbf{Ch}(R)$ is also a model category and such structure on it is similar to the former.

$f : C \to D.$ in $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ is a weak equivalence (also called quasi-isomorphism) if it is a homology isomorphism which means it induces isomorphisms between homology groups. f is a fibration if $f_n : C_n \to D_n$ for $n > 0$ are all surjective. Then cofibrations in $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ can be defined as those maps having the LLP wrt all trivial fibrations.

To be convenient, we denote the complex

$$\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow ...$$

by $R(n)$, where all terms are zero except the n-th term. And we define

$$R[n+1] := \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{id}_R} R \longrightarrow 0 \longrightarrow \cdots$$

where all terms are zero except the $(n+1)$-th and n-th terms. For $R(n)$ and $R[n+1]$ we all assume $n \geq 0$.

For any $X \in \mathrm{Ob}(\mathbf{Ch_{\geq 0}}(R))$, it's obvious to see

$$\mathrm{Hom}_{\mathbf{Ch_{\geq 0}}(R)}(R(n), X) \cong \mathrm{Hom}_R(R, Z_n(X))$$

and

$$\mathrm{Hom}_{\mathbf{Ch_{\geq 0}}(R)}(R[n+1], X) \cong \mathrm{Hom}_R(R, X_{n+1})$$

The process of the proof that $\mathbf{Ch_{\geq 0}}(R)$ is a model category is the same as that of **Top**. First we use lifting properties to characterize fibrations in $\mathbf{Ch_{\geq 0}}(R)$, then trivial fibrations. These results will play a central role in the proof of (M5).

Next we characterize the lifting property of fibrations.

Lemma 2.19. *$f : C \to D$ in $\mathbf{Ch}_{\geq 0}(R)$ is a fibration iff it has the RLP wrt all maps $0 \to R[n+1]$ for $n \geq 0$.*

Proof. The part of "⇒" is trivial, because R is projective and $C_{n+1} \to D_{n+1}$ is surjective. For any map $R \to D_{n+1}$, there exists a lifting $R \to C_{n+1}$.

"⇐". For any $y \in D_{n+1}$ there is a map $R \to D_{n+1}$ such that $1 \mapsto y$. But the map can be lifted to $R \to C_{n+1}$. Hence, there is some $x \in C_{n+1}$ such that $f_{n+1}(x) = y$, which means f is a fibration. □

Especially $0 \to R[n+1]$ for $n \geq 0$ have the LLP wrt all trivial fibrations and they are hence cofibrations. But homology groups of $R[n+1]$ are all trivial, which means these maps are trivial cofibrations.

Lemma 2.20. *The map $0 \to R(n)$ is a cofibration for all $n \geq 0$.*

Proof. If $f : C \to D$ is a trivial fibration, then there is a commutative diagram:

$$\begin{array}{ccccccc} C_1 & \longrightarrow & C_0 & \longrightarrow & H_0(C) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \cong} & & \\ D_1 & \longrightarrow & D_0 & \longrightarrow & H_0(D) & \longrightarrow & 0 \end{array}$$

Then by the four lemma or the cokernel part of the snake lemma, $\mathrm{coker}\ f_0 = 0$ and f_0 is surjective. Given a lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle f} \\ R(n) & \longrightarrow & D \end{array}$$

we need to find a lifting $R \to Z_n(C)$ to make the diagram commutative. From the fact that R is projective, we only need to prove $Z_n(C) \to Z_n(D)$ is surjective. At first we prove $f_n(B_n(C)) = B_n(D)$. $y \in D_{n+1}$, $y = f_{n+1}(x)$, $x \in C_{n+1}$ then $\partial_{n+1}(y) = \partial_{n+1}(f_{n+1}(x)) = f_n(\partial_{n+1}(x))$. Hence, $f_n(B_n(C)) = B_n(D)$. Let $y \in Z_n(D), \bar{y} \in H_n(D)$. Because $f_* : H_n(C) \to H_n(D)$ is an isomorphism, $\bar{y} = f_*(\bar{x})$ where $x \in Z_n(C)$. Hence, $f_n(x) = y + b$ where $b \in B_n(D)$. But $b = f_n(a), a \in B_n(C)$, then $y = f_n(x - a)$. □

Now we can characterize the lifting property of trivial fibrations.

Lemma 2.21. *$f : C \to D$ is a trivial fibration iff*

(1) the map f_0 is surjective in other words f has the RLP wrt $0 \to R(0)$.

(2) it has the RLP wrt all maps $g : R(n) \to R[n+1]$ for $n \geq 0$, where $g_n = \mathrm{id}_R$.

Proof. "⇒". The condition (1) is proved in the Lemma 2.20 and we only need to prove the condition (2). Given a lifting problem

$$\begin{array}{ccc} R(n) & \xrightarrow{\ x\ } & C \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ R[n+1] & \xrightarrow{\ y\ } & D \end{array}$$

where $x \in Z_n(C.)$ and $y \in D_{n+1}$ such that $\partial_{n+1}(y) = f_n(x)$, to solve this problem we need to find a suitable element $u \in C_{n+1}$ such that $\partial_{n+1}(u) = x$ and $f_{n+1}(u) = y$.

f_n is surjective for all $n \geq 0$, $f_{n+1}(z) = y$, $z \in C_{n+1}$ and $f_n(x) = \partial_{n+1} f_{n+1}(z) = f_n \partial_{n+1}(z)$. Hence $x - \partial_{n+1}(z) \in \ker f_n$ and $\partial_n(x - \partial_{n+1}(z)) = 0$. We denote $\ker f$ by K. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow D \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \to H_{n+1}(C) \xrightarrow{\cong} H_{n+1}(D) \longrightarrow H_n(K) \longrightarrow H_n(C) \xrightarrow{\cong} H_n(D.) \longrightarrow \cdots$$

and then $H_n(K)$ will be trivial. Hence $\exists k \in K_{n+1}, \partial_{n+1}(k) = x - \partial_{n+1}(z)$. We let $u = k + z \in C_{n+1}$.

"⇒". We prove f is a weak equivalence first. To prove $f_* : H_n(C) \to H_n(D)$ is surjective, we prove $Z_n(C) \to Z_n(D)$ is surjective.

$$\begin{array}{ccc} R(n) & \xrightarrow{\ 0\ } & C \\ \Big\downarrow & \overset{x}{\nearrow} & \Big\downarrow{\scriptstyle f} \\ R[n+1] & \xrightarrow{\ y\ } & D \end{array}$$

The diagram above implies $y \in D_{n+1}$, $\partial_{n+1}(y) = 0 \Leftrightarrow y \in Z_{n+1}(D)$, $x \in C_{n+1}$, $f_{n+1}(x) = y$, and $\partial_{n+1}(x) = 0 \Leftrightarrow x \in Z_{n+1}(C)$. Hence, $Z_{n+1}(c) \to Z_{n+1}(D)$ is surjective. That $Z_0(C) = C_0 \to D_0 = Z_0(D)$ is surjective is from condition (1) . If $f_*(\bar{x}') = f_*(\bar{x}'')$ where

$x', x'' \in Z_n(C)$, then $f_n(x') = f_n(x'') + \partial_{n+1}(y)$. Let $x = x' - x''$, $f_n(x) = \partial_{n+1}(y)$. There will be a commutative diagram:

$$\begin{array}{ccc} R(n) & \xrightarrow{x} & C \\ \downarrow & \overset{z}{\nearrow} & \downarrow f \\ R[n+1] & \xrightarrow{y} & D \end{array}$$

where $z \in C_{n+1}$, $x = \partial_{n+1}(z)$ which means $\bar{x} = 0$ and $\bar{x}' = \bar{x}''$. Hence f_* is an isomorphism and f is a weak equivalence. Finally we have the following exact sequence:

$$\begin{array}{ccccccc} Z_{n+1}(C) & \hookrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & Z_n(C) & \longrightarrow & H_n(C) \\ \downarrow\!\!\!\downarrow & & \downarrow f_{n+1} & & \downarrow\!\!\!\downarrow & & \downarrow \cong \\ Z_{n+1}(D) & \hookrightarrow & D_{n+1} & \xrightarrow{\partial_{n+1}} & Z_n(D) & \longrightarrow & H_n(D) \end{array}$$

By the four lemma $f_{n+1} : C_{n+1} \to D_{n+1}$ is surjective. □

From Lemma 2.19 and 2.21 we know $Fib = r(R[n+1])$ and $Fib \cap \mathcal{W} = r\big(R(0), R(n) \to R[n]\big)$. Next we can prove the axiom (M5) in $\mathbf{Ch}_{\geq 0}(R)$ using Quillen's small object argument.

Corollary 2.22. *Every chain map $f : C \to D$ has two factorizations:*

(1) $f = p \circ i$, where $p \in Fib$, and $i \in l(Fib)$ is both a monomorphism and a weak equivalence. Especially i is a trivial cofibration.

(2) $f = q \circ j$ where $q \in Fib \cap \mathcal{W}$ and $j \in l(Fib \cap \mathcal{W})$ is a monic cofibration.

Proof. Choose $\kappa = \aleph_0$.

(1). For the class of morphisms $0 \to R[n+1]$, their domains are all 0 and the representable functor $\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}(R)}(0, -)$ is zero hence commuting with arbitrary colimits, which means Theorem 2.16 is valid here. In every step $Z_i \to Z_{i+1}$ is monic and then the final morphism $X = Z_0 \to Z_\kappa$ is hence monic. And note that the homology functor H_n commutes with directed limits which is a standard exercise in any textbook about homological algebra (actually it comes from the fact that directed limits is exact). Then $H_n(Z_\kappa) \cong \operatorname*{colim}_{i \leq \kappa} H_n(Z_i)$. Moreover since H_n commutes with direct sums,

$$Z_{i+1} = Z_i \oplus (\oplus_{D_i} R[k+1]) \Rightarrow H_n(Z_{i+1}) = H_n(Z_i) \oplus (\oplus_{D_i} H_n(R[k+1])) = H_n(Z_i)$$

We conclude $H_n(Z_\kappa) = H_n(Z_0)$ and in this factorization i is a monic weak equivalence.

(2). For any chain complex X, we know

$$\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}}(R(n), X) \cong \mathrm{Hom}_R(R, Z_n(X)) \cong Z_n(X)$$

The functor $Z_n(-)$ commutes with all directed limits and hence Theorem 2.16 is valid here as well. The reason why j is monic is the same as that of i. □

(M5) has been proved and we can use it to prove (M4) that every trivial cofibration has the LLP wrt all fibrations, $Cof \cap \mathcal{W} \subseteq l(Fib)$.

Corollary 2.23.

(1) Every cofibration is a monomorphism.

(2) Every trivial cofibration has the LLP wrt all fibrations.

Proof. (1). If $j : C \to D$ is a cofibration, we prove all j_n's are injective. The chain complex

$$\cdots \longrightarrow 0 \longrightarrow C_n \xrightarrow{\mathrm{id}_{C_n}} C_n \longrightarrow 0 \longrightarrow \cdots$$

where only $(n+1)$-th and n-th terms are nontrivial, is simply denoted by $C_n[n+1]$ for convenience. Then $C_n[n+1] \to 0$ is a trivial fibration. Hence the lifting problem

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C_n[n+1] \\ {\scriptstyle j}\downarrow & \overset{g}{\nearrow} & \downarrow \\ D & \longrightarrow & 0 \end{array}$$

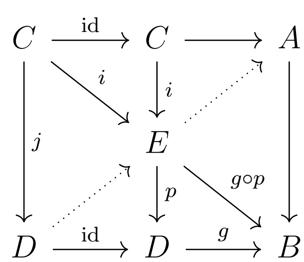
where $\alpha_{n+1} = \partial_{n+1}, \alpha_n = \mathrm{id}_{C_n}$ has a solution g. $g_n \circ j_n = \mathrm{id}_{C_n}$ therefore j_n is injective.

(2). Now we assume $j : C \to D$ is a trivial cofibration. Then it factors as $j = p \circ i : C \to E \to D$ where $i \in l(Fib)$ is a trivial cofibration and p is a trivial fibration (the first part of Corollary 2.22).

Given a lifting problem

$$\begin{array}{ccc} C & \longrightarrow & A \\ {\scriptstyle j}\downarrow & & \downarrow \\ D & \xrightarrow{g} & B \end{array}$$

where $A \to B$ is a fibration, we solve it as follows:



Hence j has the LLP wrt all fibrations. □

Now we only need to prove (M3). Since cofibrations and fibrations are all characterised by lifting properties (the former is by definition and the latter is from Lemma 2.19), from Theorem 2.10 and Corollary 2.11 they are stable under retraction. As for weak equivalences, we can use homology functors H_n and isomorphisms are stable under retraction.

Corollary 2.24. *With fibrations being maps which are surjective for $n > 0$ and weak equivalences being homology isomorphisms, $\mathbf{Ch}_{\geq 0}(R)$ is a model category.*

In the following we want to study cofibrations further.

Lemma 2.25. *A chain complex C is cofibrant iff all C_n's are projective R-modules.*

Proof. "$\Rightarrow$". Assume $A \to B$ is surjective and we obtain a trivial fibration $A[n+1] \to B[n+1]$ where

$$A[n+1] := \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots$$

and $B[n+1]$ is defined similarly. Then the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & A[n+1] \\ \big\downarrow & \nearrow & \big\downarrow \\ C & \longrightarrow & B \end{array}$$

is solved as

$$\begin{array}{ccccc} & & C_n & & \\ & \swarrow & \big\downarrow & & \\ A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Thus C_n is projective.

"$\Leftarrow$". Assume C_n's are all projective and $f : A \to B$ is a trivial fibration. $K = \ker f$, then $H_n(K) = 0$. The solution of a lifting problem can be constructed by induction using the property of projective modules. The details are left to readers. □

Corollary 2.26. *$j : C \to D$ is a cofibration iff all maps j_n for $n \geq 0$ are injective and the cokernels* $\operatorname{coker} j_n$ *are projective R-modules.*

Proof. "$\Rightarrow$". If j is a cofibration, then $D/j(C)$ will be cofibrant. Given a trivial fibration $A \to B$

$$\begin{array}{ccccc} C & \longrightarrow & 0 & \longrightarrow & A \\ \big\downarrow j & & \big\downarrow & \nearrow & \big\downarrow \\ D & \longrightarrow & D/j(C) & \longrightarrow & B \end{array}$$

Then the conclusion follows from the Lemma A.1.6.

"$\Leftarrow$". $0 \to C_n \to D_n \to D_n/C_n \to 0$ where D_n/C_n is projective, then $D_n \cong C_n \oplus D_n/C_n$. $C \to C$ and $0 \to D/j(C)$ are cofibrations. Therefore $j : C \oplus 0 \to C \oplus D/j(C) \cong D$ is a cofibration. □

Dually in the category of cochains $\mathbf{Ch}^{\geq 0}(R)$ there is a model category structure as well, in which weak equivalences are quasi-isomorphisms, cofibrations are monomorphisms for positive terms and fibrations are epimorphisms whose kernels consist of injective modules.

Therefore there are two model category structures in $\mathbf{Ch}(R)$. One is the projective model category structure and the other is the injecture one. In the projective model category of $\mathbf{Ch}(R)$, weak equivalences are quasi -isomorphisms, fibrations are epimorphisms and cofibrations are those maps having the left lifting property with respect to all trivial fibrations. Note in this case even though for every cofibrant chain complex A, A_n will be a projective R-module for all n and conversely any bounded below complex of projective R-modules is cofibrant[6], there may exist unbounded complex of projective R-modules, which isn't cofibrant. The following example comes from [Hov99a] Remark 2.3.7.

Example 2.27. Let R be the first order extension of k which means $R = k[x]/(x^2)$, where k is a field. Then $\dim_k R = 2$. Its multiplication is defined as $(a_1 + b_1 x) \cdot (a_2 + b_2 x) = a_1 a_2 + (a_1 b_2 + a_2 b_1)x$. The field k will be an R-module, with $(a + bx)c = ac$.

Now we consider the complex

$$A. := \cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow \cdots$$

where $\cdot x(a + bx) = (a + bx)x = ax$. It's obvious to see $\ker(\cdot x) = kx = \mathrm{im}(\cdot x)$. Hence A has trivial homology groups. Note that all terms in A are R, which is a free R-module, hence projective. We let $X = R(0)$ and $Y = k(0)$ centered at the 0-th position. Then there is a fibration $f : X \to Y$, $f_0(a + bx) = a$, which is an R-morphism. Given a morphism $g : A \to Y$, $g_0(a + bx) = a$ and $g_n = 0$ for $n \neq 0$. Then g is a map between chain complexes. If A is cofibrant, $0 \to A$ is trivial cofibration and the following lifting has a solution:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \overset{h}{\nearrow} & \downarrow f \\ A & \xrightarrow{g} & Y \end{array}$$

Then $h_0(1) = 1 + ax$ for some $a \in k$, and $h_0(x) = x \cdot h_0(1) = x \cdot (1 + ax) = x$. But since $h_0 \circ (\cdot x) = 0$, $h_0(x) = 0$ a contradiction! Therefore such lifting h doesn't exist and A is not cofibrant.

The injective model category structure on $\mathbf{Ch}(R)$ can be described naturally on $\mathbf{Ch}^{\bullet}(R)$ which is the category of cochain complexes. In $\mathbf{Ch}^{\bullet}(R)$ weak equivalences are quasi-isomorphisms, cofibrations are monomorphisms and fibrations are those maps having the RLP wrt all trivial cofibrations. Similarly for any fibrant object A, A_n is an injective R-module for all $n \in \mathbb{Z}$ and any bounded below complex of injective R-modules is fibrant. But it's not ture that the complex of injective R-modules is fibrant.

What's more, for an abelian category $\mathcal{A}$ if it's a Grothendieck category or has enough injective objects, $\mathbf{Ch}^{\bullet}(\mathcal{A})$ will have the injective model category structure described above. This technical fact is proved in the paper [Hov99b].

2.3 The Homotopy Theory of Model Categories

Before introducing the homotopy theory for a model category $\mathcal{M}$, we want to study the internal structure of $\mathcal{M}$ first.

Lemma 2.28. *For a weak factorization system (see Definition 2.13) $(\mathcal{F}, \mathcal{G})$, $\mathcal{F} = l(\mathcal{G})$ and $\mathcal{G} = r(\mathcal{F})$.*

[6] See [Hov99a] Lemma 2.3.6

Proof. We only need to prove $\mathcal{F} = l(\mathcal{G})$ since the other can be proved in the opposite category $\mathcal{C}^{op}$. Assume $f \in l(\mathcal{G})$ and $f = p \circ i$ where $i \in \mathcal{F}$, $p \in \mathcal{G}$. f has the LLP wrt p and from Lemma 2.7 we conclude f is a retraction of i which means $f \in \mathcal{F}$. □

Corollary 2.29. *In a model category $\mathcal{M}$,*

(1) the cofibrations are exactly those maps having the LLP wrt all trivial fibrations

(2) the trivial cofibrations are exactly those maps having the LLP wrt all fibrations

(3) the fibrations are exactly those maps having the RLP wrt all trivial cofibrations

(4) the trivial fibrations are exactly those maps having the RLP wrt all cofibrations

Proof. Apply Corollary 2.28. □

This corollary means for a given category $\mathcal{M}$ if a certain model category structure exists, then it can be characterized by $\mathcal{W}$ and Fib or $\mathcal{W}$ and Cof. But if we only know Cof and Fib, then we will also know $Cof \cap \mathcal{W}$ and $Fib \cap \mathcal{W}$ which are characterized by their lifting properties. And we can define a weak equivalence to be a composition $p \circ i$ where $i \in Cof \cap \mathcal{W}$ and $p \in Fib \cap \mathcal{W}$. Therefore $\mathcal{M}$ can also be determined by Cof and Fib. There is a refined version for Corollary 2.29.

Proposition 2.30. *In a model category $\mathcal{M}$,*

(1) a cofibration is a weak equivalence iff it has the LLP wrt all fibrations between fibrant objects

(2) a fibration is a weak equivalence iff it has the RLP wrt all cofibrations between cofibrant objects.

Proof. We only prove (1) since the second one can be proved in $\mathcal{M}^{op}$. "$\Rightarrow$" is clear and thus we prove the part of "$\Leftarrow$". Assume a cofibration $u : A \to B$ has the LLP wrt all fibrations between fibrant objects. First we choose a fibrant replacement $j : B \to B'$ where j is a trivial cofibration and B' is fibrant. Then we factor $j \circ u$ as

$$j \circ u : A \xrightarrow{i} A' \xrightarrow{p} B'$$

where i is a trivial cofibration and p is fibration which imply A is especially fibrant. Then we have the square

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle u}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[j]{} & B' \end{array}$$

$h \circ u = i$ and $p \circ h = j$ are all isomorphisms in the homotopy category $\mathrm{Ho}(\mathcal{M})$. Hence in $\mathrm{Ho}(\mathcal{M})$, h has a left inverse and a right inverse, thus an isomorphism. Then according to ??, h is a weak equivalence and therefore u is a weak equivalence. □

We also has another characterization for $\mathcal{M}$ which is useful when comparing with different models for ∞-category. We advise readers to read this proposition after reading all contents of this section, since in this proof we will use theorems below freely.

Proposition 2.31. *For a model category $\mathcal{M}$, its model category structure is determined by cofibrations and fibrant objects or fibrations and cofibrant object.*

Proof. Since two statements are dual, it's enough to prove the first one. For any object $X \in \mathrm{Ob}(\mathcal{M})$, we have the decomposition $\emptyset \to X' \xrightarrow{p_X} X$ where X' is cofibrant and p_X is a trivial fibration. For any map $u : X \to Y$ we have the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y' \\ \downarrow & u' & \downarrow p_Y \\ X' & \xrightarrow[u \circ p_X]{} & Y \end{array}$$

which permits the existence of the diagram

$$\begin{array}{ccc} X' & \xrightarrow{p_X} & X \\ u' \downarrow & & \downarrow u \\ Y' & \xrightarrow[p_Y]{} & Y \end{array}$$

Note that u is weak equivalence iff u' is a weak equivalence. Therefore weak equivalences between cofibrant objects will determine this model category structure, since from Cof we know $Fib \cap \mathcal{W}$ and $p_X,\ p_Y$ are all trivial fibrations. u' is a weak equivalence iff it's an isomorphism in $\mathrm{Ho}(\mathcal{M})$. By Yoneda's lemma, it's a weak equivalences iff for any other object A, $u'^* : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(Y', A) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A)$ is an isomorphism. But by ?? the full subcategory of $\mathrm{Ho}(\mathcal{M})$ consisting of fibrant objects is equivalent to $\mathrm{Ho}(\mathcal{M})$. Therefore we could suppose A is fibrant. But by?? this means

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A) = [X', A] = \mathrm{Hom}_{\mathcal{M}}(X', A)/\sim$$

where the equivalence relation is right homotopy or left homotopy. Here we focus on the left homotopy. But we can factor $(\mathrm{id}, \mathrm{id}) : X' \coprod X' \to X'$ as

$$X' \coprod X' \xrightarrow{i} X' \otimes I \xrightarrow{p} X'$$

where i is a cofibration and p is a trivial fibration. This factorization will make the cylinder object $X' \otimes I$ for the left homotopy relation fixed, which means this left homotopy relation is totally determined by cofibraions and trivial fibrations. Hence we conclude $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A)$ is determined. And finally weak equivalences between cofibrant objects are determined. □

For a model category $\mathcal{M}$, its *homotopy theory* or *homotopy category* is defined to be $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$ the localization respect to weak equivalences. In the following our main task is to study structures of $\mathrm{Ho}(\mathcal{M})$ in detail.

Definition 2.32. *A* ***path object*** *for $Y \in \mathrm{Ob}(\mathcal{M})$ is a commutative diagram:*

$$\begin{array}{ccc} & & Y^I \\ & \overset{s}{\nearrow} \sim & \downarrow (p_0, p_1) = p \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

where s is a weak equivalence, $\Delta = (\mathrm{id}_Y, \mathrm{id}_Y)$ and (p_0, p_1) is a fibration.

Always we simply use the symbol Y^I to denote a path object. According to the axiom (M5) of model categories, there is a natural path object for Y such that s will be a trivial cofibration.

Definition 2.33. *$f, g : X \to Y$ are two maps in $\mathcal{M}$. A **right homotopy** between f and g is a commutative diagram:*

$$\begin{array}{ccccc} & & Y^I & & \\ & \overset{h}{\nearrow} & \downarrow p & \overset{s}{\underset{\sim}{\nwarrow}} & \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xleftarrow[\Delta]{} & Y \end{array}$$

The right part of the diagram above is a path object. We denote this relation by $f \simeq_r g$.

Example 2.34. In $\mathbf{Ch}_{\geq 0}(R)$, the concept of chain homotopies is a special case of right homotopies.

Given a chain complex C , the path object is defined to be C^I such that $C^I_n = C_n \oplus C_n \oplus C_{n+1}$ for $n > 0$ and

$$C^I_0 = \{(x, y, z) \in C_0 \oplus C_0 \oplus C_1 | (x - y) + \partial_1(z) = 0\}$$

$\partial_n(x, y, z) = (\partial_n(x), \partial_n(y), (-1)^n(x - y) + \partial_{n+1}(z))$. We claim C^I is a path object.

We define a new chain complex C' as follows: $C'_n = C_n \oplus C_{n+1}$ for $n > 0$ and

$$C'_0 = \{(x, z) \in C_0 \oplus C_1 | x + \partial_1(z) = 0\}$$

$\partial_n(x, z) = (\partial_n(x), (-1)^n x + \partial_{n+1}(z))$. If $\partial_n(x, z) = 0$, then $\partial_n(x) = 0$ and $(-1)^n x + \partial_{n+1}(z) = 0$. $x = (-1)^{n+1}\partial_{n+1}(z)$, $\partial_{n+1}((-1)^{n+1}, 0) = ((-1)^{n+1}\partial_{n+1}z, (-1)^{n+1}(-1)^{n+1}z) = (x, z)$. Hence, $C' \to 0$ is a trivial fibration. There is a chain map $\alpha : C^I \to C'$, $\alpha(x, y, z) = (x-y, z)$, which is an epimorphism and $\ker \alpha \cong C$. . Then we have the following pullback diagram:

$$\begin{array}{ccc} C^I & \xrightarrow{\alpha} & C' \\ \downarrow p & & \downarrow p' \\ C \oplus C & \xrightarrow{\beta} & C \end{array}$$

where $p(x, y, z) = (x, y)$, $p'(x, z) = x$ and $\beta(x, y) = x - y$. Obviously p' is a fibration. Hence p is also a fibration. We have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{s} & C^I & \xrightarrow{\alpha} & C' & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow p & & \downarrow p' & & \\ 0 & \longrightarrow & C & \xrightarrow{\Delta} & C \oplus C & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

where $s(x) = (x, x, 0)$. There is a long exact sequence

$$\cdots \longrightarrow 0 = H_{n+1}(C') \longrightarrow H_n(C) \longrightarrow H_n(C^I) \longrightarrow 0 = H_n(C') \longrightarrow \cdots$$

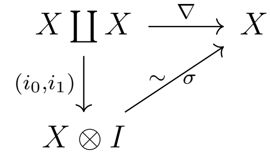
Hence $H_n(C) \cong H_n(C)$ and s is a weak equivalence.

For any two chain map $f, g : D \to C$ such that there is a usual chain homotopy $t : f \simeq g$, $t_n : D_n \to C_{n+1}$ and $\partial_{n+1} t_n + t_{n-1}\partial_n = f - g$. We define $h : C \to C^I$, $h_n(x) = (f_n(x), g_n(x), (-1)^{n+1} t_n(x))$. It's actually a chain map. $\partial h(x) = (\partial f(x), \partial g(x), (-1)^{n+1}(f(x) - g(x)) + (-1)^{n+1}\partial t(x)) = (\partial f(x), \partial g(x), (-1)^n t\partial(x)) = h\partial(x)$. $p \circ h = (f, g)$. Hence, h is a right homotopy from f to g.

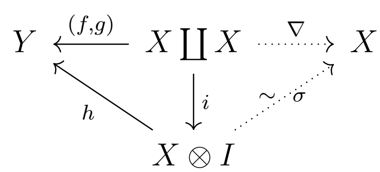
Conversely, if h is a right homotopy from f to g, we write h as (f, g, t). Because h is a chain map, $(-1)^n(f(x) - g(x)) + \partial t(x) = t\partial(x)$. Then $t'_n = (-1)^{n+1} t_n$ is a chain homotopy from f to g in the usual sense.

Another example is in **Top**, see Example 1.23. There is a dual concept for right homotopies.

Definition 2.35. *A* ***cylinder object*** *for $X \in \mathcal{M}$ is a commutative diagram:*



where $\nabla = (\mathrm{id}_X, \mathrm{id}_X)$, i is a cofibration and s is a weak equivalence. For any maps $f, g : X \to Y$, a ***left homotopy*** *$h : f \simeq_l g$ is defined to be the following commutative diagram:*



In **Top** for CW-complexes, the concept of homotopies is a special case of left homotopies.

Example 2.36. *We assume X is a CW-complex and $X \otimes I = X \times I$ where $I = [0, 1]$. Then X is a strong deformation retract of $X \times I$ and σ is especially a weak homotopy equivalence. $(X \coprod X, X \times I)$ is a relative CW-complex (see [Hat02] Theorem A.6). Hence $i : X \coprod X \hookrightarrow X \times I$ is a cofibration according to the Proposition 1.80.*

Lemma 2.37. *Let $\mathcal{M}$ be a model category,*

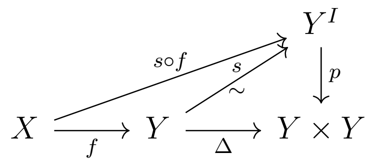
(1) if Y is fibrant, then the relation of right homotopies in $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is an equivalence relation.

(2) if X is cofibrant, then the relation of left homotopies in $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is an equivalence relation.

Proof. Axioms of a model category are all dual descriptions, which means $\mathcal{M}^{op}$ is also a model category with cofibrations becoming fibrations and fibrations becoming cofibrations. Hence we can just prove the first statement.

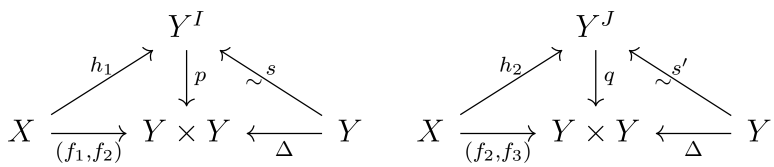
Assume Y is fibrant, which means $Y \to *$ is a fibration. In $\mathcal{M}$, for any object X, $X \times *$ is canonically isomorphic with X. Isomorphisms are both trivial cofibrations and trivial fibrations. Then the fact fibrations are preserved under products implies $X \times Y \to X$ is a fibration. Hence $pr_0, pr_1 : Y \times Y \to Y$ are fibratsions. For any path object Y^I with a fibration $p : Y^I \to Y \times Y$, $pi = pr_i \circ p$ is a fibration for $i = 0, 1$. Moreover, $p \circ s = \Delta$, $p_i \circ s = pr_i \circ \Delta = \mathrm{id}_Y$. Then p_i is a trivial fibration.

For any $f : X \to Y$, the following diagram proves $f \simeq_r f$.

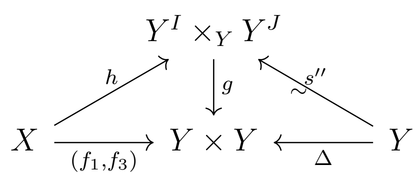


For any map $f, g : X \to Y$ and $h : f \simeq_r g$. There is an isomorphism $u = (pr_1, pr_0) : Y \times Y \to Y \times Y$. If $p : Y^I \to Y \times Y$ is the path object for the right homotopy $h : f \simeq_r g$, then $u \circ p$ is the path object for $h : g \simeq_r f$.

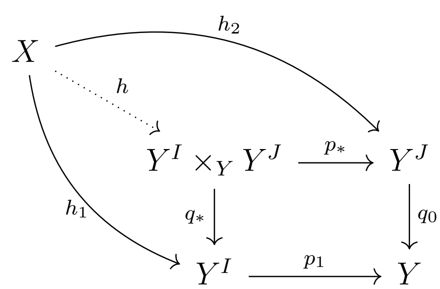
$f_1, f_2, f_3 : X \to Y$ and $h_1 : f_1 \simeq_r f_2$, $h_2 : f_2 \simeq_r f_3$.



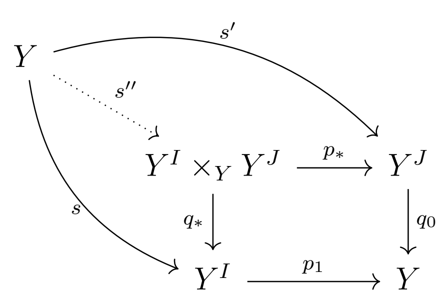
We prove there is a right homotopy:



From h_1, h_2 we know $p_0 h_1 = f_1, p_1 h_1 = f_2 = q_0 h_2, q_1 h_2 = f_3$. Then we have the following pullback diagram:



where q_0 is a trivial fibration $\Rightarrow$ q_* is also a trivial fibration.



$q_* \circ s'' = s$ is a weak equivalence $\Rightarrow s''$ is a weak equivalence. Now we only need to find the suitable g. We prove the following diagram is a pullback:

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ {\scriptstyle (q_*, q_1 p_*)}\downarrow & & \downarrow{\scriptstyle (q_0, q_1)} \\ Y^I \times Y & \xrightarrow[p_1 \times id_Y]{} & Y \times Y \end{array}$$

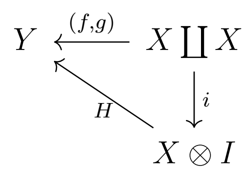
$(p_1 \times \mathrm{id}_Y) \circ (q_*, q_1 p_*) = (p_1 q_*, q_1 p_*) = (q_0 p_*, q_1 p_*) = (q_0, q_1) \circ p_*$. The diagram is commutative. Given $(u, v) : Z \to Y^I \times Y, w : Z \to Y^J$ such that $(p_1 \times \mathrm{id}_Y) \circ (u, v) = (q_0, q_1) \circ w$. Then $p_1 u = q_0 w, v = q_1 w$. From $p_1 u = q_0 w$, there is a unique $\theta : Z \to Y^I \times_Y Y^J$ such that $q_* \theta = u, p_* \theta = w$. $q_1 p_* \theta = q_1 w = v$. Hence, $(q_*, q_1 p_*) \circ \theta = (u, v)$. This proves the diagram above is actually a pullback.

We let $g = (p_0 \times \mathrm{id}_Y) \circ (q_*, q_1 p_*) = (p_0 q_*, q_1 p_*)$. $(q_*, q_1 p_*)$ is the pullback of $q = (q_0, q_1) \Rightarrow (q_*, q_1 p_*)$ is a fibration. p_0, id_Y are fibrations $\Rightarrow p_0 \times id_Y$ is a fibration. Hence g is a fibration.

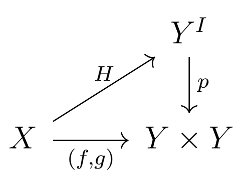
$g \circ h = (p_0 q_*, q_1 p_*) \circ h = (p_0 q_* h, q_1 p_* h) = (p_0 h_1, q_1 h_2) = (f_1, f_3)$ and $g \circ s'' = (p_0 q_*, q_1 p_*) s'' = (p_0 q_* s'', q_1 p_* s'') = (p_0 s, q_1 s') = (\mathrm{id}_Y, \mathrm{id}_Y) = \Delta$. □

Lemma 2.38.

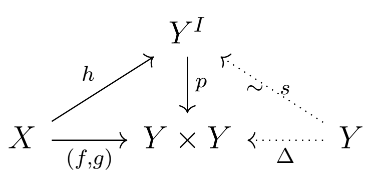
(1) If Y is fibrant, $X \otimes I$ is a fixed cylinder object for X and $f, g : X \to Y$ are right homotopic, then there is a left homotopy:



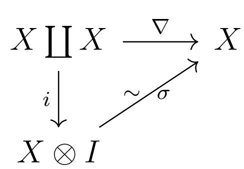
(2) If X is cofibrant, Y^I is a fixed path object for Y, and $f, g : X \to Y$ are left homotopic, then there is a right homotopy:



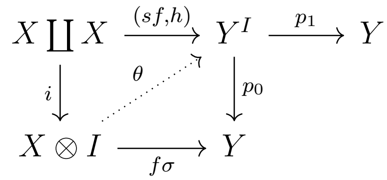
Proof. The two statements are dual and we prove the first one. Given the right homotopy $h : f \simeq_r g$:



and the fixed cylinder object:



Note that Y is fibrant $\Rightarrow p_0 : Y \times Y \to Y$ is a trivial fibration.



$p_0 \circ (sf, h) = (p_0 sf, p_0 h) = (f, f) = f\sigma i$. Let $H = p_1 \circ \theta$. $H \circ i = p_1 \theta i = p_1 \circ (sf, h) = (p_1 sf, p_1 h) = (f, g)$. ∎

Corollary 2.39. *If X is cofibrant and Y is fibrant, then for any maps $f, g : X \to Y$, $f \simeq_r g \Leftrightarrow f \simeq_l g$ for a fixed $X \otimes I \Leftrightarrow f \simeq_r g$ for a fixed $Y^I \Leftrightarrow f \simeq_l g$.*

In $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ where X is cofibrant and Y is fibrant, the two notions of right homotopy and left homotopy coincide and we simply use the symbol $f \simeq g$ to denote this homotopy relation. The homotopy class of maps between X and Y is denoted by $[X, Y] = \mathrm{Hom}_{\mathcal{M}}(X, Y)/ \sim$. If $\mathcal{M}_c$ and $\mathcal{M}_f$ are the full subcategories of $\mathcal{M}$ with all objects cofibrant and fibrant respectively, then we can define a functor $[-,-] : \mathcal{M}_c^{op} \times \mathcal{M}_f \to$ **Sets**. The fact that $[-,-]$ is well defined and actually a functor can be proved using the following lemma.

Lemma 2.40. *For any objects $X, Y \in Ob(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if $f \simeq_l g$ then for any morphism $t : Y \to Y'$, $tf \simeq_l tg$.

(2) if $f \simeq_r g$ then for any morphism $s : X' \to X$, $fs \simeq_r gs$.

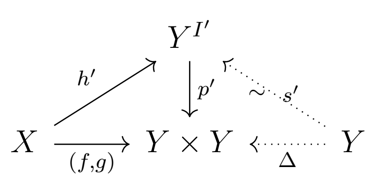
This lemma is trivial. In general, given two arbitrary morphisms $f, g : X \to Y$ and $t : Y \to Y'$, we can't conclude $tf \simeq_r tg$ from $f \simeq_r g$. But there is a weaker theorem[7], which states that in $\mathcal{M}_f$, $f \simeq_r g$, for arbitrary $t : Y \to Y'$ there exsits a trivial fibration especially a weak equivalence $u : X' \to X$ such that $tfu \simeq_r tgu$.

Lemma 2.41. *For any objects $X, Y \in \mathrm{Ob}(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if Y is fibrant and $f \simeq_l g$, then for any morphism $s : X' \to X$, $fs \simeq_l gs$.

(2) if X is cofibrant and $f \simeq_r g$ then for any morphism $t : Y \to Y'$, $tf \simeq_r tg$.

Proof. We only prove (2). Given a right homotopy $h' : f \simeq_r g$,



Decompose s' as

$$Y \xrightarrow{s} Y^I \xrightarrow{p} Y^{I'}$$

[7]See [Bro73] P423 Proposition 1.

such that s is trivial cofibration and p is a fibration. $p' \circ p$ is a fibration and Y^I is thus a path object. Since $p \circ s = s'$ is a weak equivalence, p is a trivial fibration. Then the following lifting problem has a solution since X is cofibrant.

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y^I \\ \downarrow & \overset{h}{\nearrow} & \downarrow p \\ X & \xrightarrow[h']{} & Y^{I'} \end{array}$$

Then $h : f \simeq_r g$ with the map $s : Y \to Y^I$ being a trivial cofibration. Here we simply write $p : Y^I \to Y \times Y$ for $p' \circ p$ above and use $p',\ s'$ to denote maps $Y'^I \to Y' \times Y'$ and $Y' \to Y'^I$ respectively.

$$\begin{array}{ccccc} & & Y & \xrightarrow{s't} & Y'^I \\ & & s\downarrow & \overset{k}{\nearrow} & \downarrow p' \\ X & \xrightarrow[h]{} & Y^I & \xrightarrow[(t,t)\circ p]{} & Y' \times Y' \end{array}$$

$k \circ h : tf \simeq_r tg.$ □

For any two objects $X,\ Y \in \mathrm{Ob}(\mathcal{M})$, $\pi^l(X,Y)$ (resp. $\pi^r(X,Y)$) denotes the quotient set $\mathrm{Hom})_{\mathcal{M}}(X,Y)/\sim$ where the equivalence relation is generated by left (resp. right) homotopy. $f \sim_l g$ in $\mathrm{Hom}_{\mathcal{M}}(X,Y)$ if there is a long sequence of left homotopy connecting them, which means $f \simeq_l f_1 \simeq_l f_2 \simeq_l \cdots \simeq_l g$.

Corollary 2.42. *For any objects $X, Y \in \mathrm{Ob}(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if Y is fibrant, then the composition $\pi^l(X', X) \times \pi^l(X,Y) \to \pi^l(X',Y)$ is well defined.

(2) if X is cofibrant, then the composition $\pi^r(X,Y) \times \pi^r(Y,Y') \to \pi^r(X,Y')$ is well defined.

We can define the homotopy category $\pi\mathcal{M}_c$ (resp. $\pi\mathcal{M}_f$) to be the quotient category of $\mathcal{M}_c$ (resp. $\mathcal{M}_f$) where the equivalence relation is the right (resp. left) homotopy relation and $\mathrm{Hom}_{\pi\mathcal{M}_c}(X,Y) = \pi^r(X,Y)$. Then from Corollary 2.42 this definition is well defined. $\pi\mathcal{M}_{cf}$ is in the usual sense with $\mathrm{Hom}_{\pi\mathcal{M}_{cf}}(X,Y) = [X,Y]$ since $X,\ Y$ are both cofibrant and fibrant.

Next we prove the modern version of Whitehead's theorem whose classical version says a weak homotopy equivalence between CW-complexes is homotopy equivalence.

Theorem 2.43 (Whitehead)**.** *If X, Y are both cofibrant and fibrant, then every weak equivalence $f : X \to Y$ is a homotopy equivalence.*

Proof. According to (M5) of model categories, $f : X \xrightarrow{i} Z \xrightarrow{p} Y$ where i is a trivial cofibration and p is trivial fibration. We prove Z is both cofibrant and fibrant first.

Let $A \to B$ be a trivial cofibration:

$$\begin{array}{ccccc} A & \longrightarrow & Z & \xrightarrow{p} & Y \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & * & \longrightarrow & * \end{array}$$

where Y is fibrant and p is trivial fibration. Hence Z is fibrant. Dually, Z is cofibrant.

If p and i are homotopy equivalences, there are $q : Y \to Z, j : Z \to X$ such that $pq \simeq \mathrm{id}_Y, qp \simeq \mathrm{id}_Z, ij \simeq \mathrm{id}_Z, ji \simeq \mathrm{id}_X$. Then

$$fjq = pijq \simeq p\mathrm{id}_Z q \simeq pq \simeq \mathrm{id}_Y$$

and

$$jqf = jqpi \simeq j\mathrm{id}_Z i \simeq ji \simeq \mathrm{id}_X$$

which means f is a homotopy equivalence as well. Now we need to prove every trivial fibration is a homotopy equivalence and it's dual to prove every trivial cofibration is a homotopy equivalence. Hence we can just assume f is a trivial fibration.

Since Y is cofibrant,

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow f \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

then $fg = \mathrm{id}_Y$, and we only need to prove $gf \simeq \mathrm{id}_X$.

Given a cylinder object,

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ i \big\downarrow & \overset{\sim\ \sigma}{\nearrow} & \\ X \otimes I & & \end{array}$$

there exists the following homotopy $h : gf \simeq_l \mathrm{id}_X$ since f is a trivial fibration and i is cofibration.

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(gf, \mathrm{id}_X)} & X \\ i \big\downarrow & \overset{h}{\nearrow} & \big\downarrow f \\ X \otimes I & \xrightarrow[f\sigma]{} & Y \end{array}$$

where $f \circ (gf, \mathrm{id}_X) = (fgf, g) = (f, f) = f\sigma i$. □

In **Top** every space is fibrant and every CW-complex is cofibrant. Hence the classical Whitehead's theorem tells us every weak homotopy equivalence between CW-complexes is actually a homotopy equivalence. And in $\mathbf{Ch}_{\geq 0}(R)$, every chain complex is fibrant and every chain complex with every term projective is cofibrant. Then, we can conclude two projective resolutions of a given R-module is chain homotopic.

The converse of Whitehead's theorem is also ture. If f is a homotopy equivalence then f is a weak equivalence. We will prove it later.

Now for every object X in $\mathcal{M}$, we find a suitable object RQX which is both cofibrant and fibrant and is weak equivalent with X. At first, we use the axiom (M5) to decompose $\emptyset \to X$ as $\emptyset \longrightarrow QX \xrightarrow{p_X} X$ such that QX is cofibrant and p_X is a trivial fibration. Then we decompose $QX \to *$ as $QX \xrightarrow{j_X} RQX \longrightarrow *$ such that j_X is a trivial cofibration and

RQX is fibrant. In fact, because QX is cofibrant, RQX is cofibrant automatically. Hence RQX is both cofibrant and fibrant.

According to the descriptions above, we can find maps $X \xleftarrow{p_X} QX \xrightarrow{j_X} RQX$ for every $X \in \mathcal{M}$ such that:

(1) p_X is a trivial fibration, j_X is a trivial cofibration, QX is cofibrant and RQX is both cofibrant and fibrant.

(2) if X is cofibrant, $QX = X, p_X = \mathrm{id}_X$. And if QX is fibrant, $RQX = QX, j_X = \mathrm{id}_{QX}$.

From the condition (2), we can conclude $Q(RQX) = RQX$, $Q(QX) = QX$, $(RQ)(RQX) = RQX$, $(RQ)(QX) = RQX$.

If in our model category the weak factorization system is functorial, then we can find functorial replacement functors Q and R. But without this assumption the functorial property is only up to homotopy.

Lemma 2.44. *For any map $f : X \to Y$, there will exist a commutative diagram:*

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

Moreover f_2 is unique up to homotopy.

Proof. Since QX is cofibrant and p_Y is a trivial fibration,

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & \overset{f_1}{\nearrow} & \downarrow{\scriptstyle p_Y} \\ QX & \xrightarrow[f \circ p_X]{} & Y \end{array}$$

Since RQY is fibrant and j_X is a trivial cofibration,

$$\begin{array}{ccc} QX & \xrightarrow{j_Y \circ f_1} & RQY \\ {\scriptstyle j_X}\downarrow & \overset{f_2}{\nearrow} & \downarrow \\ RQX & \longrightarrow & * \end{array}$$

Now we prove the uniqueness. If there is another commutative diagram (f_1', f_2'), we first prove $f_1 \simeq_l f_1'$. Given a cylinder object

$$\begin{array}{ccc} QX \coprod QX & \xrightarrow{\nabla} & QX \\ {\scriptstyle I}\downarrow & \overset{\sim\ \sigma}{\nearrow} & \\ QX \otimes I & & \end{array}$$

since i is a cofibration and p_Y is a trivial fibration,

$$\begin{array}{ccccc} QX\coprod QX & \xrightarrow{(f_1,f_1')} & QY & \xrightarrow{j_Y} & RQY \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p_Y} & & \\ QX\otimes I \xrightarrow{\sigma} QX & \xrightarrow{fp_X} & Y & & \end{array} \tag{6}$$

where $p_Y \circ (f_1, f_1') = (p_Y f_1, p_Y f_1') = (fp_X, fp_X) = fp_X\sigma i$. From $f_1 \simeq_l f_1'$, we conclude $j_Y f_1 \simeq_l j_Y f_1'$. Hence $f_2 j_X \simeq_l f_2' j_X$. Since QX is cofibrant and RQY is fibrant, $f_2 j_X \simeq_r f_2' j_X$ and we have the following homotopy diagram:

$$\begin{array}{ccc} & & RQY^I \\ & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ QX & \xrightarrow[(f_2 j_X, f_2' j_X)]{} & RQY \times RQY \end{array}$$

and since p is a fibration and j_X is a trivial cofibration

$$\begin{array}{ccc} QX & \xrightarrow{h} & RQY^I \\ {\scriptstyle j_X}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle p} \\ RQX & \xrightarrow[(f_2, f_2')]{} & RQY \times RQY \end{array}$$

Hence $f_2 \simeq f_2'$. □

According to the Lemma 2.44, we can define a functor $RQ : \mathcal{M} \to \pi\mathcal{M}_{cf}$ such that $X \mapsto RQX$ and $f \mapsto [RQf] = [f_2]$. From the proof above we know that the functor $Q : \mathcal{M} \to \pi\mathcal{M}_c$ is also well defined, since according to Lemma 2.38 (2) the left homotopy $f_1 \simeq_l f_1'$ can be changed into a right homotopy. Similarly there is a functor $R : \mathcal{M} \to \pi\mathcal{M}_f$ as well. Therefore we say the factorization system in a model category is functorial up to homotopy.

Now we can use the funtor $RQ : \mathcal{M} \to \pi\mathcal{M}_{cf}$ to obtain the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$. Objects of $\mathrm{Ho}(\mathcal{M})$ are the same as $\mathcal{M}$ and

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\pi\mathcal{M}_{cf}}(RQX, RQY) = [RQX, RQY]$$

From the uniqueness of f_2 up to homotopy, there is a functor

$$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M}),\ \gamma(X) = X,\ \gamma(f) = [RQf] = [f_2]$$

and the inclusion functor $\bar{\gamma} : \pi\mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M})$ is fully faithful and essentially surjective hence an equivalence between categories.

What's more, Whitehead's theorem tells us that if f is weak equivalence, then $\gamma(f)$ is an isomorphism. In the following, we will prove γ is actually a localization functor and $\mathrm{Ho}(\mathcal{M})$ is the category of fractions of $\mathcal{M}$ with respect to the set $\mathcal{W}$ of weak equivalences. Moreover $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence.

Proposition 2.45. *If we assume X is cofibrant and Y is fibrant, then there is a bijection $[X, Y] \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = [RQX, RQY]$.*

Proof. Y is fibrant $\Rightarrow QY$ is fibrant. Then $QX = X$, $RQX = RX$, $RQY = QY$, $p_X = \mathrm{id}_X$,$j_Y = \mathrm{id}_{QY}$.

$\gamma : \mathrm{Hom}_{\mathcal{M}}(X, Y) \to [RQX, RQY] = [RX, QY]$. Given any map $f_2 : RX \to QY$, let $f_1 = f_2 \circ j_X$ and $f = p_Y \circ f_1$, then $\gamma(f) = [f_2]$ which means γ is surjective.

Next, we prove γ factors through $[X, Y]$. If $f, g : X \to Y$ and $f \simeq g$, we prove $f_2 \simeq g_2$. Since X is cofibrant and p_Y is a trivial fibration:

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & \nearrow^{f_1}\ \nearrow_{g_1} & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f/g]{} & Y \end{array}$$

where $f_1,\ g_1$ are liftings of $f,\ g$ respectively.

Given a left homotopy $h : f \simeq_l g$, we have a lifting

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(f_1, g_1)} & QY \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p_Y} \\ X \otimes I & \xrightarrow[h]{} & Y \end{array}$$

where $h \circ i = (f, g)$. Hence $f_1 \simeq g_1$. The same process will imply $f_2 \simeq f_2$. Then γ can be factored as

$$\mathrm{Hom}_{\mathcal{M}}(X, X) \longrightarrow [X, Y] \xrightarrow{\gamma'} [RQX, RQY] = [RX, QY]$$

Conversely if $f_2 \simeq g_2$, then $f_1 = f_2 \circ j_X \simeq g_2 \circ j_X = g_1$ and $f = p_Y \circ f_1 \simeq p_Y g_1 = g$. Hence γ is a bijection. □

Theorem 2.46. *$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is the localization functor for the category of fractions $\mathcal{M}[\mathcal{W}^{-1}]$, which means for any functor $F : \mathcal{M} \to \mathcal{D}$ taking weak equivalences to isomorphisms, there exists a unique functor $F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ such that $F_* \circ \gamma = F$.*

Proof. Assume $f, g : X \to Y$ in $\mathcal{M}$, if $f \simeq_r g$ or $f \simeq_l g$, then $F(f) = F(g)$. The proofs are the same. Hence we can just assume $h : f \simeq_r g$. $s : Y \to Y^I$ is a weak equivalence hence $F(s)$ an isomorphism. $p_i \circ s = \mathrm{id}_Y \Rightarrow F(p_i) = F(s)^{-1}$ and

$$f = p_0 \circ h, g = p_1 \circ h \Rightarrow F(f) = F(g) = F(s)^{-1}F(h)$$

On objects F_* is easily defined, for $\mathrm{Ho}(\mathcal{M})$ has the same objects as $\mathcal{M}$. Now suppose $[f] \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = [RQX, RQY]$ where $f : RQX \to RQY$.

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ & & & & \downarrow{\scriptstyle f} \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

where p_X, p_Y, j_X, j_Y are all weak equivalences. Thus

$$\begin{array}{ccccc} F(X) & \xleftarrow{\sim} & F(QX) & \xrightarrow{\sim} & F(RQX) \\ & & & & \downarrow{\scriptstyle F(f)} \\ F(Y) & \xleftarrow{\sim} & F(QY) & \xrightarrow{\sim} & F(RQY) \end{array}$$

We define

$$F_*([f]) = F(p_Y) \circ F(j_Y)^{-1} \circ F(f) \circ F(j_X) \circ F(p_X)^{-1}$$

If $[f] = [g]$, $f \simeq g$ then $F(f) = F(g)$. Hence F_* is well defined. It's obvious to see F_* is actually a functor. Now we need to prove $F_* \circ \gamma = F$. Given a map $f : X \to Y$, $\gamma(f) = [RQf] = [f_2]$.

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

Take F in this diagram and we obtain:

$$\begin{array}{ccccc} F(X) & \xleftarrow{\sim} & F(QX) & \xrightarrow{\sim} & F(RQX) \\ {\scriptstyle F(f)}\downarrow & & {\scriptstyle F(f_1)}\downarrow & & {\scriptstyle F(f_2)}\downarrow \\ F(Y) & \xleftarrow{\sim} & F(QY) & \xrightarrow{\sim} & F(RQY) \end{array}$$

It's obvious to see $F_*\gamma(f) = F_*([f_2]) = F(f)$. Then we should prove F_* is unique.

Given $f : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$ that is $[f] \in [RQX, RQY]$. Since

$$RQ(QX) = RQX, \quad RQ(RQX) = RQX$$

f can also represent an element of $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(QX, QY)$ and of $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(RQX, RQY)$. Consider the following diagram

$$\begin{array}{ccccc} QX & \xleftarrow{\mathrm{id}} & Q(QX) = QX & \xrightarrow{j_X} & RQ(RQX) = RQX \\ {\scriptstyle p_X}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xleftarrow[p_X]{} & QX & \xrightarrow[j_X]{} & RQX \end{array}$$

Hence $\gamma(p_X) = [\mathrm{id}_{RQX}]$.

$$\begin{array}{ccccc} QX & \xleftarrow{\mathrm{id}} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle j_X}\downarrow & & {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ RQX & \xleftarrow[\mathrm{id}]{} & RQX & \xrightarrow[\mathrm{id}]{} & RQX \end{array}$$

Then $\gamma(j_X) = [\mathrm{id}_{RQX}]$. Therefore we have the following commutative diagram in the category $\mathrm{Ho}(\mathcal{M})$:

$$\begin{array}{ccccc} X & \xleftarrow{\gamma(p_X)} & QX & \xrightarrow{\gamma(j_X)} & RQX \\ {\scriptstyle [f]}\downarrow & & {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [f]} \\ Y & \xleftarrow[\gamma(p_Y)]{} & QY & \xrightarrow[\gamma(j_Y)]{} & RQY \end{array}$$

$[f] \in [RQX, RQY]$, if $f : RQX \to RQY$ in $\mathcal{M}$, then $\gamma(f) = [f]$. $F = F_* \circ \gamma$ forces

$$F_*([f]) = F(p_Y) \circ F(j_Y)^{-1} \circ F(f) \circ F(j_X) \circ F(p_X)^{-1}$$

for $[f] : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$. Hence F_* is unique. □

We can also consider the localization for $\mathcal{M}_c$ and $\mathcal{M}_f$ with respect to weak equivalences, and it's obvious to see $\mathcal{M}_c[\mathcal{W}^{-1}]$ (resp. $\mathcal{M}_c[\mathcal{W}^{-1}]$) is equivalent to the full subcategory of $\mathrm{Ho}(\mathcal{M})$ consisting of cofibrant objects (resp. fibrant objects), using the same method above. We can check the universal property of localization for subcategories $\mathrm{Ho}(\mathcal{M}_c)$ and $\mathrm{Ho}(\mathcal{M}_f)$ of $\mathrm{Ho}(\mathcal{M})$ directly. Moreover due to the existence of the factorization system in $\mathcal{M}$, $\mathrm{Ho}(\mathcal{M}_c)$ and $\mathrm{Ho}(\mathcal{M}_f)$ are all actually equivalent to $\mathrm{Ho}(\mathcal{W})$.[8] Finally we have the following commutative diagram:

$$\begin{array}{ccc} \pi\mathcal{M}_c & \xrightarrow{\bar{\gamma}_c} & \mathrm{Ho}(\mathcal{M}_c) \\ \uparrow & & \downarrow{\scriptstyle\cong} \\ \pi\mathcal{M}_{cf} & \xrightarrow[\sim]{\bar{\gamma}} & \mathrm{Ho}(\mathcal{M}) \\ \downarrow & & \uparrow{\scriptstyle\cong} \\ \pi\mathcal{M}_f & \xrightarrow{\bar{\gamma}_f} & \mathrm{Ho}(\mathcal{M}_f) \end{array} \tag{7}$$

Note that $\bar{\gamma}_c$ (resp. $\bar{\gamma}_f$) comes from the restriction of γ to $\mathcal{M}_c$ (resp. $\mathcal{M}$) and from the following lemma.

Lemma 2.47. *Let $F : \mathcal{M} \to \mathcal{D}$ carry weak equivalences into isomorphisms. If $f \simeq_l g$ or $f \simeq_r$ then $F(f) = F(g)$.*

Proof. Since $f \simeq_l g$ and $f \simeq_r g$ are dual, we may assume $f \simeq_l g$. Let $h : X \otimes I \to Y$ be the left homotopy of f and g.

$$\begin{array}{ccccc} Y & \xleftarrow{(f,g)} & X \coprod X & \xrightarrow{\nabla} & X \\ & {\scriptstyle h}\nwarrow & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \sim\ \sigma} & \\ & & X \otimes I & & \end{array}$$

σ is a weak equivalence $\Rightarrow F(\sigma)$ is an isomorphism. Then for $i_0, i_1 : X \to X \otimes I$, $F(i_0) = F(i_1) = F(\sigma)^{-1}$. Hence

$$F(f) = F(h \circ i_1) = F(h) \circ F(\sigma)^{-1} = F(g)$$

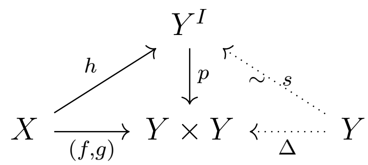
[8] This is a theorem in [Qui67] chapter 1, p1.13, theorem 1.

□

Finally we want prove for the localization functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, if $\gamma(f)$ is an isomorphism, then f is a weak equivalence. Since homotopy equivalences are sent to isomorphisms via γ, this theorem will imply every homotopy equivalence is a weak equivalence. We give a proof here following [Rie20] which is different from that in [Qui67] and [GoJ99].[9]

Lemma 2.48. *$f, g : X \to Y$ are arbitrary two maps in $\mathcal{M}$. If $f \simeq_r g$ or $f \simeq_l g$, then f is a weak equivalence iff g is a weak equivalence.*

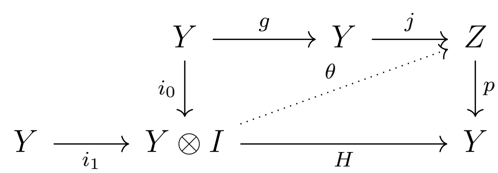
Proof. Proofs for the two conditions are the same. Hence we assume $f \simeq_r g$ and f is a weak equivalence.



In this diagram, p_i is weak equivalence for $i = 0, 1$. $f = p_0 \circ h \Rightarrow h$ is a weak equivalence. Hence $g = p_1 \circ h$ is a weak equivalence. □

Lemma 2.49. *If X, Y are both cofibrant and fibrant, $f : X \to Y$ in $\mathcal{M}$ such that $\gamma(f)$ is an isomorphism, then f is weak equivalence.*

Proof. That $\gamma(f)$ is an isomorphism means f is homotopy equivalence between X and Y. We decompose f as $X \xrightarrow{j} Z \xrightarrow{p} Y$ where j is a trivial cofibration, p is a fibration and Z is both cofibrant and fibrant. We only need to prove p is a weak equivalence. Since f is a homotopy equivalence, there exists $g : Y \to X$ such that $gf \simeq \mathrm{id}_X, fg \simeq \mathrm{id}_Y$. If $H : fg \simeq id_Y$ is the left homotopy for a cylinder object $Y \otimes I$, then we have the following diagram:



where $pjg = fg = Hi_0$, i_0 is a trivial cofibration and p is a fibration.

We let $k = \theta \circ i_1 : Y \to Z$. $pk = p\theta i_1 = Hi_1 = \mathrm{id}_Y$. $\theta i_0 = jg$. Hence $\theta : jg \simeq_l k$. According to Whitehead's theorem, j is a homotopy equivalence. Then there is the homotopy inverse $q : Z \to X$ such that $qj \simeq \mathrm{id}_X, jq \simeq \mathrm{id}_Z$.

From

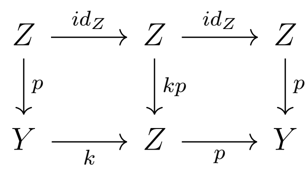
$$jq \simeq \mathrm{id}_Z,\ jg \simeq k,\ gf \simeq \mathrm{id}_X$$

we conclude

$$kp \simeq kpjq = kfq \simeq jgfq \simeq jq \simeq id_Z$$

[9]In [Qui67] p5.2 Lemma 1 is central for the proof, but the part of (3) $\Rightarrow$ (2) is much more complicated. It uses the calculation of homotopies. Gievn a left homotopy $h_1 : f \simeq_l g$ you should know the concrete description of $h_1^{-1} : g \simeq_l f$ and the compositions of homotopies $h_2 * h_1 : f \simeq_l h$, where $h_2 : g \simeq_l h$. To know these, you should be familiar with the dual proof of the Lemma A.2.9. All these proofs can also be found in [GoJ99] Chapter II section 1, Poposition 1.14 and Lemma 1.15.

Hence kp is a weak equivalence.



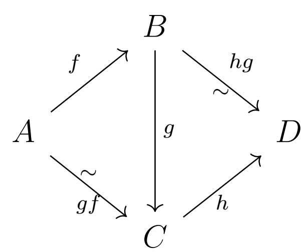
p is the retraction of kp. Hence p is a weak equivalence. □

Corollary 2.50. *For any $f : X \to Y$ in $\mathcal{M}$, if $\gamma(f)$ is an isomorphism, then f is a weak equivalence.*

Proof. That $\gamma(f)$ is an isomorphism means f_2 is a homotopy equivalence. According to the Lemma 2.49, f_2 is a weak equivalence. Hence f is also a weak equivalence. □

This corollary tells us that the class $\mathcal{W}$ of weak equivalences coincides with the class of morphisms which are inverted by the functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. This fact tells us that in a model category weak equivalences satisfy a property called *two out of six*, which is similar to the property called two out of three appearing in the definition of model categories.

Corollary 2.51. *Given a commutative diagram*



If gf and hg are weak equivalences, then so are $f,\ g,\ h,\ hgf$.

Proof. Take the functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ to this diagram. $\gamma(gf), \gamma(hg)$ are isomorphisms. Then $\gamma(g)$ has a right inverse $\gamma(f) \circ \gamma(gf)^{-1}$ and a left inverse $\gamma(hg)^{-1} \circ \gamma(h)$. Hence $\gamma(g)$ is an isomorphism. The Corollary 2.50 tells us g will be a weak equivalence. Then f, h, hgf are all weak equivalences. □

3 Derived Category

In this chapter, we will describe $\mathbf{Ch}(R)[\mathcal{W}^{-1}]$ where $\mathcal{W}$ denotes the class of quasi-isomorphisms in the category of chain complexes, using the method of *calculus of fractions*. And then, we will describe the general Verdier quotient in triangulated categories. Actually Quillen's work on model categories is motivated by the work of Kan on simplicial sets and of Verdier on derived categories. Now to see the motivation of calculus of fractions, we begin this section with the localization of rings.

3.1 The Localization of Rings

For convenience, we consider the localization of commutative rings first. For a moment we assume a ring is a commutative ring with a unit.

If R is a commutatuve ring and S is a set consisting of some elements in R, the localization ring R_S of R with respect to S is important in the study of commutative rings theory. The localization R_S with the canonical ring map $\tau : R \to R_S$ satisfies a universal property that is :

$$\begin{array}{ccc} R & \xrightarrow{\tau} & R_S \\ {\scriptstyle f}\downarrow & \swarrow_{\exists!\, f'} & \\ R' & & \end{array}$$

the diagram above is commutative, where $\forall s \in S$, $f(s)$ is invertible in R'.

From the universal property of the localization, it's easy to see if S is replaced by the multiplicative set S' generated by S, then $R_S \cong R_{S'}$. Hence, we may assume S is a multiplicative set that means $1 \in S$ and $\forall s, t \in S \Rightarrow st \in S$. In general, there are two ways to construct R_S. The one is classical and systematic while the other is constructive.

Classically, we can define $R_S = \{\frac{r}{s} | r \in R, s \in S\}$, where $\frac{r}{s} = \frac{r'}{s'}$ iff $\exists t \in S$, $(rs' - r's)t = 0$ in R. Then we can define addition and multiplication on R_S making it an actual commutative ring. $\frac{r_1}{s_2} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$, $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$. There are some things should be checked. First the relation on R_S is actually an equivalencce relation. Second, the addition and multiplication defined above are well defined. Third, all these actually define a commutative ring. We leave these to readers.

Theorem 3.1. *$\tau : R \to R_S$, $r \mapsto \frac{r}{1}$ satisfies the universal property of localization for commutative rings.*

Proof. If f' exists, then $f'(\frac{r}{s}) = f(r)f(s)^{-1}$. Hence it will be unique. We only need to prove f' is well defined. If $\frac{r}{s} = \frac{r'}{s'}$, then $\exists t \in S$, $(rs' - r's)t = 0$ in R. $[f(r)f(s') - f(r')f(s)]f(t) = 0$ in R'. But $f(t)$ is a unit. Therefore $f(r)f(s') = f(r')f(s)$ and then $f(r)f(s)^{-1} = f(r')f(s')^{-1}$. f' is well defined. □

From the Theorem 3.1 above, it's easy to see when $R \neq 0$, $R_S = 0$ iff $0 \in S$. Hence we may assume $0 \notin S$ in general.

Now we state the second way to describe R_S. We know, for any R-algebra R', $R' \cong R[X_i; i \in I]/\sim$. We give this polynomial form of R_S. Suppose S is generated by $\{s_i | i \in I\}$, which means $s_i \in S$ and for any element $s \in S$, $s = s_{i_1} \cdot ... \cdot s_{i_n}$. Obvioully, S is generated by itself.

Proposition 3.2. $R_S \cong R[X_i; i \in I]/(1 - s_i X_i; i \in I)$.

Proof. We only need to prove $\tau : R \to R[X_i; i \in I]/(1 - s_i X_i; i \in I)$, $r \mapsto \bar{r}$ satisfies the universal property in the diagram (B.1). If f' exists, then $f'(\bar{r}) = f(r)$. From $\bar{s}_i \bar{X}_i = 1$, $f'(\bar{X}_i) = f(s_i)^{-1}$. Then f' will be unique. We prove it's well defined next. If $\sum_u \bar{a}_u \bar{X}^u = 0$, then $\sum_u a_u X^u = \sum_i g_i(X)(1 - s_i X_i)$, where $g_i(X) \in R[X_i; i \in I]$. Therefore $f'(\sum_u \bar{a}_u \bar{X}^u) = \sum_i r'_i(1 - f(s_i) f(s_i)^{-1}) = 0$. f' is well defined. □

According to the Proposition B.1.1, every element of R_S has the form $\frac{\tau(r)}{\tau(s)}$. This fact depends heavily on the commutativity of R. Otherwise, the addition and multiplication are hard to describe. Note $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}$ is actually $r_1 s_1^{-1} r_2 s_2^{-1}$. If R is not commutative we may not change the position of s_1^{-1} and r_2.

Next we will describe the localization for non-commutative rings, which is much more complicated than the case where rings are commutative.[10]

Fact 3.3. In the case of commutative rings $\tau : R \to R_S$ has the following important properties:

(1) Every element of R_S has the form $\tau(r)\tau(s)^{-1}$, where $s \in S$.

(2) $\ker \tau = \{r \in R | \exists s \in S, rs = 0\}$.

(1′) Every element of R_S has the form $\tau(s)^{-1}\tau(r)$, where $s \in S$.

(2′) $\ker \tau = \{r \in R | \exists s \in S, sr = 0\}$.

Even if R is non-commutative, R_S always exists. But in general R_S won't satisfy the properties of (1) and (2) or $(1')$ and $(2')$ above.

Theorem 3.4. *Assume R is a noncommutative ring with a unit and S is a multiplicative set. Then R_S always exists and if $R \neq 0$, $R_S = 0$ iff $0 \in S$.*

Proof. For any $s \in S$, we add s^* to the set R such that $s^* s = 1, ss^* = 1$. Then we have a new set $X = R \coprod \{s^*; s \in S\}$. Every element of R_S is a sum of words, such as $rs^*r' + s'^*r''s''^*$. It's obvious to see R_S will satisfy the universal property of the localization with respect to S.

There is another method to prove this theorem. A ring R is actually an additive category, with only one element $*$ such that $\mathrm{Hom}(*, *) = R$ with composition being multiplication. Then from the Remark 2.2, the localization $*[S^{-1}]$ of the category $*$ with respect to the morphism set S exists, whose morphism set is actually R_S. □

In the following we always assume S is a multiplicative set such that $0 \notin S$.

In general, R_S has very bad behaviors. Even though R is a noncommutative domain with $S = R - \{0\}$, $\tau : R \to R_S$ may not be injective and R_S may not be a division ring. All these difficulties lie in the fact that we can't describe the form of elements in R_S clearly and always there is a complicated relation on R_S. To obtain such clear description like the

[10]See [Lam99] for the application of the localization technique to non-commutative algebras.

property 1. above, we need to propose some additional structures for the multiplicative set S, so that $\tau : R \to R_S$ will satisfy the properies $(1)$and (2) or $(1')$ and $(2')$.

Definition 3.5. *$R' \neq 0$ is said to be a* ***right ring of fractions*** *with respect to $S \subseteq R$, if there is a ring homomorphism $\varphi : R \to R'$ such that:*

1. *$\forall s \in S$, $\varphi(s)$ is a unit in R'.*
2. *Every element of R' has the form $\varphi(r)\varphi(s)^{-1}$.*
3. *$\ker \varphi = \{r \in R | \exists s \in S,\ rs = 0\}$.*

If R' is a right ring of fractions, then $\forall s \in S, r \in R$, $sR \cap rS \neq \emptyset$. In fact $\varphi(s)^{-1}\varphi(r) = \varphi(a)\varphi(t)^{-1}$, $\varphi(rt) = \varphi(sa)$. Hence $\exists s' \in S$, $(rt - sa)s' = 0$. Then $rts' = sas'$, where $ts' \in S$. This condition is called **right Ore**.

If $s' \in S, r \in R$, $s'r = 0$, then $\varphi(s')\varphi(r) = 0$. But $\varphi(s')$ is a unit in R'. Hence $\varphi(r) = 0$. $\exists s \in S$, $rs = 0$. The condition that $\exists s' \in S, s'r = 0 \Rightarrow \exists s \in S, rs = 0$ is called **right reversible**. Similarly, there are left Ore and left reversible conditions.

Theorem 3.6. *R has a right ring of fractions with respect to S iff S is right Ore and right reversible.*

Proof. The part of "$\Rightarrow$" has been proved. We assume S is right Ore and right reversible. In the following, we construct a new ring RS^{-1}, which is in fact satisfies the universal property of the localization with respect to S.

Due to the form rs^{-1}, we start with $R \times S$ and define a relation on it.

$(a, s) \sim (a', s')$ iff $\exists b, b' \in R$, $(ab, sb) = (a'b', s'b')$ in $R \times S$. Note $sb = s'b'$ in S. The definition is intuitive, for $\frac{a}{s} = \frac{ab}{sb} = \frac{a'b'}{s'b'} = \frac{a'}{s'}$. We prove it's an equivalence relation.

It's obvious to see $(a, s) \sim (a, s)$ and $(a, s) \sim (a', s') \Rightarrow (a', s') \sim (a, s)$. Therefore we assume $(a, s) \sim (a', s'), (a', s') \sim (a'', s'')$. Then $(ab, sb) = (a'b', s'b'), (a'c, s'c) = (a''c', s''c')$ in $R \times S$. $s'c, s'b' \in S$, according to the right Ore condition $s'cS \cap s'b'R \neq \emptyset$, there exist $t \in S, r \in R$ such that $s'ct = s'b'r \in s'cS \cap s'b'R \subseteq S$. Since S is reversible and $s' \in S$, $\exists t' \in S$, $ctt' = b'rt'$. Then $s'b'rt' \in S$.

Thus, $(abrt', sbrt') = (a'b'rt', s'b'rt') = (a'ctt', s'ctt') = (a''c'tt', s''c'tt')$, which means $(a, s)brt' = (a'', s'')c'tt'$ in $R \times S$.

From the definition of the equivalence relation above, for any $b \in R$ such that $sb \in S$, $(a, s) \sim (ab, sb)$. We define $RS^{-1} = R \times S/\sim$ and the equivalence class of (a, s) is denoted by as^{-1} or $\frac{a}{s}$. Next we define the addition and multiplication on RS^{-1}.

Assume $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in RS^{-1}$. Since $s_1S \cap s_2R \neq \emptyset$, $\exists t \in S, r \in R$, $s_1t = s_2r \in S$. Then $\frac{a_1}{s_1} = \frac{a_1t}{s_1t}$ and $\frac{a_2}{s_2} = \frac{a_2r}{s_2r}$. Therefore, we define $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1t + a_2r}{s_1t = s_2r}$. We should prove it's well defined.

If $(a_1, s_1) \sim (a_1', s_1')$ and $(a_2, s_2) \sim (a_2', s_2')$. $\frac{a_1'}{s_1'} + \frac{a_2'}{s_2'} = \frac{a_1't' + a_2'r'}{s_1't' = s_2'r'}$. We need to prove $(a_1t + a_2r, s_1t = s_2r) \sim (a_1't' + a_2'r', s_1't' = s_2'r')$. Now we have $(a_1t, s_1t) \sim (a_1, s_1) \sim (a_1', s_1') \sim (a_1't', s_1't')$. Similiarly, $(a_2r, s_2r) \sim (a_2'r', s_2'r')$. In $R \times S$, there are following equations:

$$\begin{cases} (a_1tb, s_1tb) = (a_1't'b', s_1't'b') \\ (a_2rc, s_2rc) = (a_2'r'c', s_2'r'c') \end{cases}$$

the right Ore condition tells us $s_1tbS \cap s_2rcR \neq \emptyset$, then $\exists x \in S, y \in R$, $s_1tbx = s_2rcy = s_2rbx \in S$. Since $s_2r \in S$, $\exists u \in S$, $cyu = bxu$. Hence, $s_1tbxu = s_1't'b'xu = s_1tcyu = s_2rcyu = s_2'r'c'yu = s_1't'c'yu$. $s_1't' \in S$, then $\exists v \in S, b'xuv = c'yuv$.

From $\begin{cases} cyuv = bxuv \\ c'yuv = b'suv \end{cases}$, we have the following equations:

$$\begin{cases} (a_1tbxuv, s_1tbxuv) = (a_1't'b'xuv, s_1't'b'xuv) \\ (a_2rcyuv, s_2rcyuv) = (a_2'r'c'yuv, s_2'r'c'yuv) \end{cases}$$

$(a_1t + a_2r, s_1t = s_2r)bxuv = (a_1't' + a_2'r', s_1't' = s_2'r')b'xuv$. $(RS^{-1}, +)$ is an additive group.

Note though we use the right Ore condition $s_1S \cap s_2R \neq \emptyset$, we only need the fact $s_1t = s_2r \in S$, no matter $t \in S$ or not. Thus, $(R, +)$ is commutative. The natural map $\tau : R \to RS^{-1}$, $\tau(a) = \frac{a}{1}$, is then an additive group homomorphism. $\tau(a) = 0 \Leftrightarrow (a, 1) \sim (0, 1) \Leftrightarrow (as, s) = (0, s) \Leftrightarrow as = 0$. $ker\ \tau = \{a \in R | \exists s \in S,\ as = 0\}$.

In the following, we define the multiplication on RS^{-1}. Note, we can't define $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}$ as $\frac{a_1a_2}{s_1s_2}$ directly, because we should write $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}$ as $a_1s_1^{-1} \cdot a_2s_2^{-1}$ formally. To have the form rt^{-1} we should change the position of s_1^{-1} and a_2 to some degree. But the right Ore condition tells us $a_2S \cap s_1R \neq \emptyset$, $\exists t \in S, r \in R$, $a_2t = s_1r$, which intuitively means $s_1^{-1}a_2 = rt^{-1}$and then $a_1s_1^{-1} \cdot a_2s_2^{-1} = a_1rt^{-1}s_2^{-1} = a_1r(s_2t)^{-1}$. Thus we define $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1r}{s_2t}$. Again, we should prove it's well defined.

Assume $(a_1, s_1) \sim (a_1', s_1')$ and $(a_2, s_2) \sim (a_2', s_2')$. $\frac{a_1'}{s_1'} \cdot \frac{a_2'}{s_2'} = \frac{a_1'r'}{s_2't'}$, where $a_2't' = s_1'r'$. We need to prove $(a_1r, s_2t) \sim (a_1'r', s_2't')$. From the assumption above, we have equations in $R \times S$:

$$\begin{cases} (a_1b, s_1b) = (a_1'b', s_1'b') \\ (a_2c, s_2c) = (a_2'c', s_2'c') \end{cases}$$

$\frac{a_1b}{s_1b} \cdot \frac{a_2c}{s_2c} = \frac{a_1'b'}{s_1'b'} \cdot \frac{a_2'c'}{s_2'c'} = \frac{a_1bx}{s_2cy} = \frac{a_1'b'x}{s_2'c'y}$, where $s_1bx = a_2cy, x \in R, y \in S$.

Therefore we need to prove $(a_1bx, s_2cy) \sim (a_1r, s_2t)$, with $\begin{cases} s_1bx = a_2cy \\ s_1r = a_2t \end{cases}$. Since $t \in S$, $tR \cap cyS \neq \emptyset$, $\exists u \in R, v \in S$, $tu = cyv$. Then $a_2tu = s_1ru = a_2cyv = s_1bxv$. $s_1 \in S \Rightarrow \exists w \in S, ruw = bxvw$. Then $(a_1bx, s_2cy)vw = (a_1bxvw, s_2cyvw) = (a_1ruw, s_2tuw) = (a_1r, s_2t)uw$. Similarly, $(a_1'b'x, s_2'c'y) \sim (a_1'r', s_2't')$. It's done.

It's obvious to see $\frac{1}{1}$ is the unit of $(RS^{-1}, \cdot)$. Now, we should prove $(RS^{-1}, +, \cdot)$ is a ring. $(\frac{a_1}{s_1} + \frac{a_2}{s_2}) \cdot \frac{b}{t} = \frac{a_1r_1 + a_2r_2}{s_1r_1 = s_2r_2} \cdot \frac{b}{t} = \frac{(a_1r_1 + a_2r_2)y}{tx}$, where $bx = s_1r_1y = s_2r_2y$. But $\frac{a_1}{s_1} \cdot \frac{b}{t} + \frac{a_2}{s_2} \cdot \frac{b}{t} = \frac{a_1r_1y}{tx} + \frac{a_2r_2y}{tx}$. $\frac{b}{t} \cdot (\frac{a_1}{s_1} + \frac{a_2}{s_2})$ is similar.

$\tau : R \to RS^{-1}$ is actually a ring homomorphism, hence a right ring of fractions with $\tau(s)^{-1} = \frac{1}{s}$ for $s \in S$. □

Corollary 3.7. *$\tau : R \to RS^{-1}$ defined above is actually a localization map with respect to S.*

Proof. Given a ring homomorphism $f : R \to R'$ such that for all $s \in S$, $f(s)$ is a unit in R'. If f' exists, $f'(\frac{a}{s}) = f(a)f(s)^{-1}$ is unique. Hence we should prove it's well defined. If

$\frac{a}{s} = \frac{a'}{s'}$, $(a,s) \sim (a',s')$,$\exists b, b' \in R$, $(ab, sb) = (a'b', s'b')$ in $R \times S$. Then $f(s)(b) = f(s')f(b') \Rightarrow f(b) = f(s)^{-1}f(s')f(b')$. $f(ab) = f(a)f(s)^{-1}f(s')f(b') = f(a')f(b')$. Since $f(s'b')$ and $f(s')$ are units, $f(b')$ is a unit. Then $f(a)f(s)^{-1}f(s') = f(a') \Rightarrow f(a)f(s)^{-1} = f(a')f(s')^{-1}$. f' is well defined. □

From the proofs above, R_S is a right ring of fractions iff S is right Ore and right reversible. If S is left Ore and left reversible as well, $S^{-1}R \cong R_S \cong RS^{-1}$.

The work on the localization of non-commutative rings was due to Ore in the early 1930s, when he considered the case where R is a domain and $S = R - \{0\}$.

3.2 Calculus of Fractions

In fact according to the proof of the Theorem 3.4, the theory of the localization of rings can be covered by the theory of the localization of categories. Therefore it may be possible to use the technique from the localization of rings to study the localization of categories since the composition of morphisms is just like the multiplication in a non-commutative ring and both of them are non-commutative.

Definition 3.8. *Given a category $\mathcal{C}$ and a morphism set S closed under compositions, we say S is **right Ore** and **right reversible** or S is a **multiplicative system**, if it satisfies the following conditions:*

1. *(right Ore): Given morphisms $f \in \mathrm{Mor}(\mathcal{C})$, $s \in S$ with cod(f) = cod(s), there exist $t \in S$ and $g \in \mathrm{Mor}(\mathcal{C})$ such that $f \circ t = s \circ g$, which means $f \circ S \cap s \circ \mathrm{Mor}(\mathcal{C}) \neq \emptyset$.*

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow[f]{} & Z \end{array}$$

2. *(right reversible): Given morphisms $f, g : X \rightrightarrows Y$ in $\mathcal{C}$, if there is a morphism $t : Y \to Y' \in S$ such that $t \circ f = t \circ g$, then there is a morphism $s : X' \to X$ such that $f \circ s = g \circ s$.*

$$X' \xrightarrow{s} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{t} Y'$$

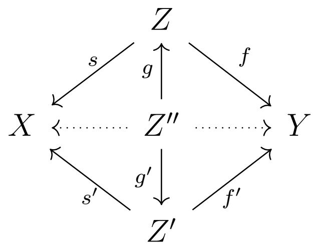
Definition 3.9. *Given a category $\mathcal{C}$ and a morphism set S closed under compositions, we say $(\mathcal{C}, S)$ **admits a calculus of right fractions** if there is a functor $\tau : \mathcal{C} \to \mathcal{D}$ such that:*

1. *$\forall f \in S$, $\tau(f)$ is an isomorphism in $\mathcal{D}$.*
2. *Every morphism in $\mathcal{D}$ has the form $\tau(f)\tau(s)^{-1}$ where $s \in S$.*
3. *$\tau(f) = \tau(g)$ iff $\exists s \in S$, $f \circ s = g \circ s$.*

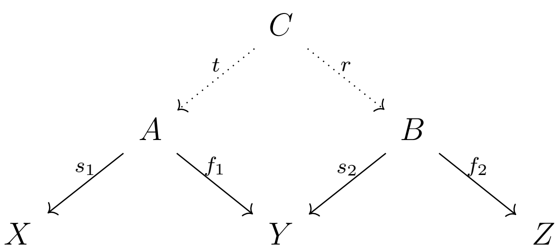
Theorem 3.10. *$(\mathcal{C}, S)$ admits a calculus of right fractions iff S is right Ore and right reversible.*

Proof. The part of "⇒" is similar to the case of rings. Given $f \in \mathrm{Mor}(\mathcal{C})$, $s \in S$ with $cod(f) = cod(s)$, since $\tau(s)^{-1}\tau(f) = \tau(g)\tau(t)^{-1}$, $\tau(ft) = \tau(gs)$, then $\exists u \in S$, $ftu = gsu$ where $tu \in S$. If $s' \circ f = s' \circ g$, where $s' \in S$, then $\tau(s')\tau(f) = \tau(s')\tau(g)$. Since $\tau(s')$ is an isomorphism, $\tau(f) = \tau(g)$ and $\exists v \in S$, $fv = gv$.

Next we prove the part of "⇐". Similar to the case of rings as well, we construct a new category $\mathcal{C}S^{-1}$ which is in fact $\mathcal{C}[S^{-1}]$. $\mathcal{C}S^{-1}$ has objects the same as $\mathcal{C}$. We define $\mathrm{Hom}_{\mathcal{C}S^{-1}}(X,Y) = \{(f,s)|do(f) = do(s), f \in \mathrm{Mor}(\mathcal{C}), s \in s\}/\sim$. $(f,s) \sim (f',s')$ iff $\exists g, g' \in \mathrm{Mor}(\mathcal{C})$, $(fg, sg) = (f'g', s'g')$ in $\mathrm{Mor}(\mathcal{C}) \times S$. The relation is actually an equivalence relation, which is left to readers.



We write the equivalence class of (f,s) as fs^{-1}. Now we define the compositions. Given morphims (f_1, s_1) and (f_2, s_2):



we define $f_2 s_2^{-1} \circ f_1 s_1^{-1} = (f_2 r)(s_1 t)^{-1}$. We leave readers to prove it's well defined. Now we should prove with the composition above $\mathcal{C}S^{-1}$ is actually a category. $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}_X^{-1} \Rightarrow fs^{-1} \circ \mathrm{id}_X = fs^{-1}$ and $\mathrm{id}_Y \circ fs^{-1} = fs^{-1}$. $(f_3 s_3^{-1} \circ f_2 s_2^{-1}) \circ f_1 s_1^{-1} = f_3 r (s_2 t)^{-1} \circ f_1 s_1^{-1} = f_3 r r'(s_1 t')^{-1}$ where $s_3 r = f_2 t, s_2 t r' = f_1 t'$. Then $f_3 s_3^{-1} \circ (f_2 s_2^{-1} \circ f_1 s_1^{-1}) = f_3 s_3^{-1} \circ f_2 t r'(s_1 t')^{-1} = f_3 r r'(s_1 t')^{-1}$.

Finally we prove $\tau : \mathcal{C} \to \mathcal{C}S^{-1}, \tau(f) = f\mathrm{id}^{-1}$ is a calculus of right fractions. Obviously, τ is a functor, making morphisms $s \in S$ being isomorphisms in $\mathcal{C}S^{-1}$with $\tau(s)^{-1} = \mathrm{id}s^{-1}$. If $\tau(f) = \tau(g) \Leftrightarrow (f, \mathrm{id}) \sim (g, \mathrm{id}) \Leftrightarrow \exists s \in S$, $(fs, s) = (gs, s) \Leftrightarrow \exists s \in S, fs = gs$. □

Corollary 3.11. *$\mathcal{C}S^{-1}$ defined above is the localization of $\mathcal{C}$ with respect to S.*

Proof. Given a functor $F : \mathcal{C} \to \mathcal{D}$ making morphisms $s \in S$ being isomorphisms in $\mathcal{D}$, if $F' : \mathcal{C}S^{-1} \to \mathcal{D}$ such that $F' \circ \tau = F$ exists, then $F'(fs^{-1}) = F(f)F(s)^{-1}$ is unique. We prove it's well defined. If $fs^{-1} = gt^{-1} \Leftrightarrow (fr, sr) = (gr', tr')$ in $\mathrm{Mor}(\mathcal{C}) \times S$. $F(s)F(r) = F(t)F(r')$, $F(r) = F(s)^{-1}F(t)F(r')$. Then $F(f)F(r) = F(f)F(s)^{-1}F(t)F(r') = F(g)F(r')$. Since $F(tr')$ and $F(t)$ are isomorphisms, $F(r')$ is an isomorphism. Hence $F(f)F(s)^{-1} = F(g)F(t)^{-1}$. F' is well defined. □

If $(\mathcal{C}, S)$ admits a calculus of left fractions as well, then $S^{-1}\mathcal{C} \cong \mathcal{C}[S^{-1}] \cong \mathcal{C}S^{-1}$. In the Section B.4, we will prove for any abelian category $\mathcal{A}$, $(K(\mathcal{A}), \mathcal{W})$ admits a calculus of left and right fractions using the method of triangulated categories, where $K(\mathcal{A})$ is the homotopy category of chain complexes and $\mathcal{W}$ denote the set of quasi-isomorphisms.

In the previous section, we consider model categories and we have proved the Whitehead's theorem and its converse. $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, where $\mathcal{M}$ is a model category, $\gamma(f)$ is an isomorphisms in $\mathrm{Ho}(\mathcal{M})$ iff f is a weak equivalence. There is a similar but weaker theorem for calculus of fractions.

Theorem 3.12. *If $(\mathcal{C}, S)$ admits a calculus of left and right fractions and S satisfies the two out of three property, then via $\tau : \mathcal{C} \to \mathcal{C}[S^{-1}]$, $\tau(g)$ is an isomorphism in $\mathcal{C}[S^{-1}]$ iff $\exists f, h \in \mathrm{Mor}(\mathcal{C})$ such that $gf, hg \in S$.*

Proof. The part of "$\Leftarrow$" is obvious. $\tau(g)$ will have a left inverse and a right inverse hence an isomorphism. We now assume $g : X \to Y$ is an isomorphism in $\mathcal{C}[S^{-1}]$. $fs^{-1} = t^{-1}h : Y \to X$ is the inverse of g in $\mathcal{C}[S^{-1}]$. Then $(gf, s) \sim (\mathrm{id}_Y, \mathrm{id}_Y)$, $\exists v \in S, u \in \mathrm{Mor}(\mathcal{C})$, $(gfu, su) = (v, v)$ in $\mathrm{Mor}(\mathcal{C}) \times S$. Since $v = su, s \in S$ and S satisfies the property of two out of three, $u \in S$. Then $v = gfu, u \in S \Rightarrow gf \in S$. Similarly, from $(t, hg) \sim (\mathrm{id}_X, \mathrm{id}_X)$, we conclude $hg \in S$. □

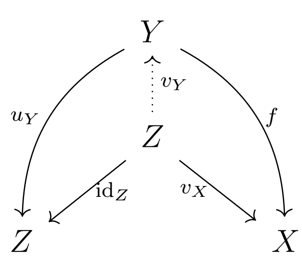
Even though this theorem is weaker than that in model categories, if S satisfies the property of two out of six, then $\tau(g)$ is an isomorphism iff $g \in S$. And this works on homotopical categories[11] in which the class of weak equivalences satisfies the two out of six property.

We now try to compare $\mathcal{C}$ and $\mathcal{C}S^{-1}$ via the calculus of right fractions.

Lemma 3.13. *If $(\mathcal{C}, S)$ admits a calculus of right fractions and Z is a zero object in $\mathcal{C}$, then via $\tau : \mathcal{C} \to \mathcal{C}S^{-1}$, $\tau(Z)$ is a zero object in $\mathcal{C}S^{-1}$.*

Proof. For any $X \in \mathrm{Ob}(\mathcal{C}S^{-1})$ there are maps $u_X : X \to Z$ and $v_X : Z \to X$ coming from $\mathcal{C}$. We need to prove they are unique in $\mathcal{C}S^{-1}$ as well. Given any $X \xleftarrow{s} Y \xrightarrow{u_Y} Z$ representing a map $X \to Z$ in $\mathcal{C}S^{-1}$, then $(u_Y, s) \sim (u_X, \mathrm{id}_X)$, $(u_Y, s) = (u_X s, s)$, since Z is terminal and $u_X s : Y \to Z$ is unique.

On the other hand, given any $Z \xleftarrow{u_Y} Y \xrightarrow{f} X$ representing a map $Z \to X$ in $\mathcal{C}S^{-1}$, the following diagram implies $(f, u_Y) \sim (v_X, \mathrm{id}_Z)$, since Z is initial.



□

Next we prove an important theorem.

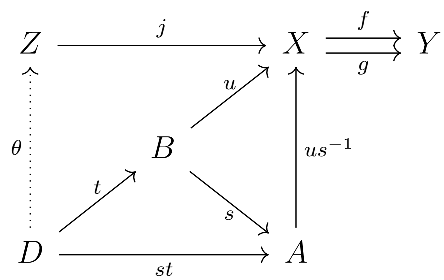
[11]See [Riel14] for more properties of homotopical category which is a generalization of model categories.

Theorem 3.14. *If $(\mathcal{C}, S)$ admits a calculus of right fractions, then $\tau : \mathcal{C} \to \mathcal{C}S^{-1}$ preserves finite limits. Dually, if $(\mathcal{C}, S)$ admits a calculus of left fractions, then $\tau : \mathcal{C} \to S^{-1}\mathcal{C}$ preserves finite colimits.*

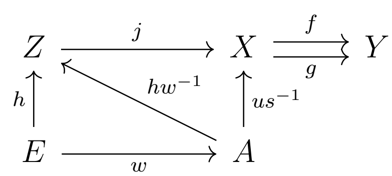
Proof. It's necessary to only prove the theorem when $(\mathcal{C}, S)$ admits a calculus of right fractions. We prove τ preserves equalizers first and then prove it preserves finite products.

$$Z \xrightarrow{j} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

The diagram above is an equalizer in $\mathcal{C}$, and we should prove it's an equalizer in $\mathcal{C}S^{-1}$ as well. Given any morphisms $A \xleftarrow{s} B \xrightarrow{u} X$ representing $us^{-1} : A \to X$ in $\mathcal{C}S^{-1}$ such that $fus^{-1} = gus^{-1}$, then $fu = gu$ in $\mathcal{C}S^{-1}$ since s^{-1} is an isomorphism. Thus $\exists t \in S$, $fut = gut$ in $\mathcal{C}$. Since Z is an equalizer, $\exists! \theta$, $j\theta = ut$. Then $\theta(st)^{-1} : A \to Z$ is the solution.

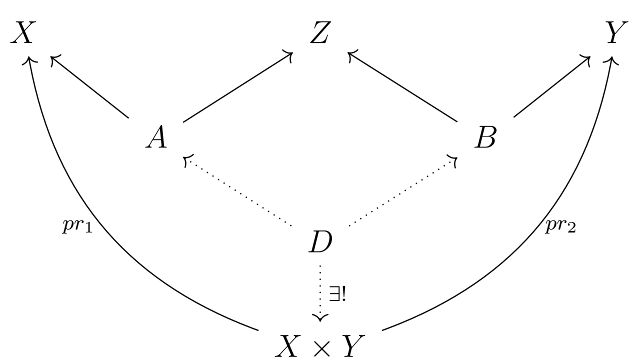


If there is any other solution:

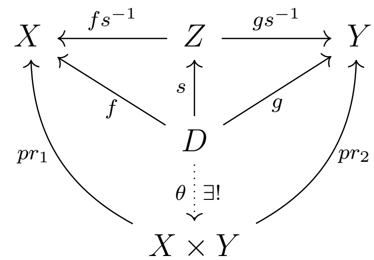


$jhw^{-1} = us^{-1} = j\theta(st)^{-1} \Rightarrow (jh, w) \sim (j\theta, st) \Rightarrow (jhb, wb) = (j\theta b', stb')$ in $\mathrm{Mor}(\mathcal{C}) \times S$. Since j is the equalizer, hence monic, $hb = \theta b' \Rightarrow (hb, wb) = (\theta b', stb') \Rightarrow hw^{-1} = \theta(st)^{-1}$ in $\mathcal{C}S^{-1}$.

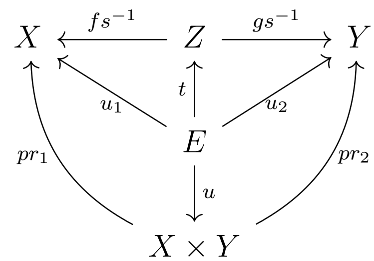
The proof that τ preserves finite products is more complicated. If $X \times Y$ is a product in $\mathcal{C}$, we prove it's the product in $\mathcal{C}S^{-1}$ as well. Given any morphisms $Z \longleftarrow A \longrightarrow X$ and $Z \longleftarrow B \longrightarrow Y$ representing morphisms $Z \to X$ and $Z \to Y$ in $\mathcal{C}S^{-1}$. According to the right Ore condition, we have the following diagram:



Therefore we can delete A and B in the diagram above:

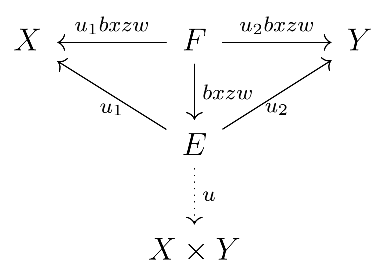


where $pr_1 \circ \theta = f$, $pr_2 \circ \theta = g$. Then $Z \xleftarrow{s} D \xrightarrow{\theta} X \times Y$ is the solution. If there is another solution:



where $u_i = pr_i \circ u$, $i = 1, 2$, $u_1 t^{-1} = fs^{-1}$, $u_2 t^{-1} = gs^{-1}$. Now we should prove $(u, t) \sim (\theta, s)$. We already know that $(u_1, t) \sim (f, s)$ and $(u_2, t) \sim (g, s)$, $\begin{cases} (u_1 b, tb) = (fb', sb') \\ (u_2 c, tc) = (gc', sc') \end{cases}$. Since $tb \circ \mathrm{Mor}(\mathcal{C}) \cap tc \circ S \neq \emptyset$, there exist $x \in \mathrm{Mor}(\mathcal{C}), y \in S$ such that $tbx = tcy \in S$. But $t \in S$, $\exists z \in S, bxz = cyz$. Then $tbxz = sb'xz = tcyz = sc'yz$. Since $s \in S$, $\exists w \in S$, $b'xzw = c'yzw$.

$(u_1 bxzw, tbxzw) = (fb'xzw, sb'xzw)$, $(u_2 cyzw, tcyzw) = (gc'yzw, sc'yzw)$. Therefore $u_1 bxzw$ and $u_2 cyzw = u_2 bxzw$ give a unique $v : F \to X \times Y$ in $\mathcal{C}$, where $F = \mathrm{dom}(w)$.



This diagram implies $v = u \circ bxzw$. Similiarly, $v = \theta \circ b'xzw$. Hence $(ubxzw, tbxzw) = (\theta b'xzw, tbxzw) = (\theta b'xzw, sb'xzw)$, which means $(u, t) \sim (\theta, s)$. θs^{-1} is unique. $\square$

This theorem tells us that $\mathcal{C}S^{-1}$ will have enough limits if $\mathcal{C}$ has enough limits.

Corollary 3.15. *If $(\mathcal{C}, S)$ admits a calculus of right fractions and $\mathcal{C}$ is an additive category, then $\mathcal{C}S^{-1}$ is an additive category as well and the localization functor τ is additive.*

Proof. For any $f_1 s_1^{-1}, f_2 s_2^{-1} : X \to Y$ in $\mathcal{C}S^{-1}$, $s_1 \circ \mathrm{Mor}(\mathcal{C}) \cap s_2 \circ S \neq \emptyset$, $s_1 r = s_2 t$, then we can define $\frac{f_1}{s_1} + \frac{f_2}{s_2} = \frac{f_1 r + f_2 t}{s_1 r = s_2 t} = (f_1 r + f_2 t)(s_1 r)^{-1}$. Similar to the case of rings, it's well defined, $\mathrm{Hom}_{\mathcal{C}S^{-1}}(X,Y)$ is an abelian group and τ is additive. According to the Lemma 3.13 and the Theorem 3.14, there exist the zero object and direct sums in $\mathcal{C}S^{-1}$. □

There is a similar theorem for abelian categories.

Theorem 3.16. *If $(\mathcal{C}, S)$ admits a calculus of right and left fractions and $\mathcal{C}$ is an abelian category, then $\mathcal{C}[S^{-1}]$ is an abelian category as well.*

Proof. $C[S^{-1}]$ is additive accdoring to the corollary above. Theorem 3.14 tells us that the localization functor τ preserves finite limits and finite colimits, and kernels and cokernels are in fact equalizers and coequalizers respectively. Then τ preserves kernels and cokernels. And isomorphisms don't affect kernels and cokernels. Hence every morphism in $\mathcal{C}[S^{-1]}]$ has the kernel and cokernel. In fact given $fs^{-1} = t^{-1}g : X \to Y$ in $\mathcal{C}[S^{-1}]$, $\ker(fs^{-1}) \cong \ker g$ and $\mathrm{coker}(fs^{-1}) \cong \mathrm{coker}\, f$.

$$\begin{array}{ccccccc} A & \xrightarrow{\ker g} & X & \xrightarrow{\quad fs^{-1}=t^{-1}g \quad} & Y & \xrightarrow{\mathrm{coker}\ f} & B \\ & & \downarrow & & \uparrow & & \\ & & \mathrm{coim}\ g & \xrightarrow{\sim} \ \mathrm{coim}\ f \ \xrightarrow{\sim} & \mathrm{im}\ f & & \end{array}$$

where $\mathrm{coim}\ g = \mathrm{coker}(\ker\ g)$ and $\mathrm{im}\ f = \ker(\mathrm{coker}\ f)$. $\mathrm{coim}\ g \cong \mathrm{coim} f$ is an isomorphism in $\mathcal{C}[S^{-1}]$ since f and g are isomorphic, while $\mathrm{coim}\ f \xrightarrow{\sim} \mathrm{im}\ f$ is due the fact that $\mathcal{C}$ is an abelian category. □

3.3 The Localization of Subcactegories

Now we consider the localization for full subcategories. We assume $(\mathcal{C}, S)$ admits a calculus of right fractions and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. Then $S \cap \mathrm{Mor}(\mathcal{D})$ will be a morphism set in $\mathcal{D}$. If $S \cap \mathrm{Mor}(\mathcal{D})$ is right Ore and right reversible as well in $\mathcal{D}$, the $\mathcal{D}[(S \cap \mathrm{Mor}(\mathcal{D}))^{-1}]$ is simply denoted by $\mathcal{D}S^{-1}$.

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}S^{-1} \\ \downarrow & & \\ \mathcal{C} & & \\ \downarrow & \swarrow & \\ \mathcal{C}S^{-1} & & \end{array}$$

There will exist a natural fnctor $\mathcal{D}S^{-1} \to \mathcal{C}S^{-1}$. That $\mathcal{D}$ is a *localizing subcategory* is equivalent to say:

1. Given $X, Y \in \mathrm{Ob}(\mathcal{D})$, $\forall (f, s) \in (\mathrm{Mor}(\mathcal{C}), S)$ representing a morphism $X \to Y$ in $\mathcal{C}S^{-1}$, there exists $(g, t) \in (\mathrm{Mor}(\mathcal{D}), \mathrm{Mor}(\mathcal{D}) \cap S)$ such that $(f, s) \sim (g, t)$ in $\mathcal{C}$.
2. For any two $(f, s), (g, t) \in (\mathrm{Mor}(\mathcal{D}), \mathrm{Mor}(\mathcal{D}) \cap S)$, if $(f, s) \sim (g, t)$ in $\mathcal{C}$, then $(f, s) \sim (g, t)$ in $\mathcal{D}$.

Theorem 3.17. *If $\mathcal{D}$ is a localizing subcategory of $(\mathcal{C}, S)$ and $\forall X \in \mathcal{C}$, $\exists f : X \to Y$ or $g : Z \to X$ such that $f, g \in S$ and $Y, Z \in \mathrm{Ob}(\mathcal{D})$, then $\mathcal{D}S^{-1} \cong \mathcal{C}S^{-1}$. If moreover $\mathrm{Mor}(\mathcal{D}) \cap S$ consists of isomorphisms, then $\mathcal{D} \cong \mathcal{D}S^{-1} \cong \mathcal{C}S^{-1}$.*

Proof. The natural functor $\mathcal{D}S^{-1} \to \mathcal{C}S^{-1}$ will be essentially surjective. □

Example 3.18. We assume R is a commutative ring with S a multiplicative set. Σ denotes the class of morphisms $f : A \to B$ in the category of R-modules, such that $f_* : A \otimes_R R_S \to B \otimes_R R_S$ is an isomorphism in $R_S - \mathbf{Mod}$. Obviously Σ contains all isomorphisms, since $- \otimes_R R_S$ is a functor. Next we prove Σ is right Ore and right reversible.

Since R_S is flat and the tensor product is left adjoint, $- \otimes_R R_S$ preserves finite limits and arbitrary colimits. Given morphisms $A \xrightarrow{f} B \xleftarrow{g} C$ where $g \in \Sigma$, we let D be the pullback and then $D \otimes_R R_S$ will be the pullback as well. But g_* is an isomorphism, and this implies $D \to A \in \Sigma$. Hence Σ is right Ore.

Given morphisms $f, g : A \to B$ and $t : C \to A$, if $ft = gt$, $t \in \Sigma$, then $f_* = g_*$. We choose $s : B \to D$ is the equalizer of f, g and obviously $fs = gs$. Since tensor products preserve colimits, s_* will be the equalizer of f_*, g_*, which is an isomorphism. Thus Σ is right reversible.

Then $(R - \mathbf{Mod}, \Sigma)$ admits a calculus of right fractions and similarly it admits a calculus of left fractions. We now consider the full subcategory $R_S - \mathbf{Mod} \subseteq R - \mathbf{Mod}$. For any R_S-module A, using the universal property of the localization with respect to S, it's obvious to see $A \otimes_R R_S \cong A$. Hence $\Sigma \cap \mathrm{Mor}(R_S - \mathbf{Mod})$ consists of isomorphisms. And it's trivial to prove $R_S - \mathbf{Mod}$ satisfies the two conditions of localizing subcategory. You can check this using the fact that an R-map $f : A \to B$ with B an R_S-module factors uniquely through the R_S-map $A \otimes_R R_S \to B \cong B \otimes_R R_S$. Moreover, for any R-module M, there is a natural map $M \to M \otimes_R R_S$, which induce the isomorphism $M \otimes_R R_S \xrightarrow{\sim} M \otimes_R R_S \otimes_R R_S \cong M \otimes_R R_S$, hence belonging to Σ. Finally, $R_S - \mathbf{Mod} \cong R - \mathbf{Mod}[\Sigma^{-1}]$.

Sometimes the two conditions stated above are cumbersome to check. In the following we give a more flexible criterion.

Theorem 3.19. *Given a localization system $(\mathcal{C}, S)$ which is right Ore and right reversible, and a full subcategory $\mathcal{D}$ if for all $s : X \to Y \in S, Y \in \mathrm{Ob}(\mathcal{D})$, there exists a morphism $f : Z \to X$ with $Z \in \mathrm{Ob}(\mathcal{D})$ such that $s \circ f : Z \to X \to Y \in S \cap \mathrm{Mor}(\mathcal{D})$, then $\mathcal{D}$ is a localizing subcategory of $\mathcal{C}$.*

Proof. We first prove $\mathcal{D}$ is right Ore and right reversible.

$$\begin{array}{ccccc} E & \xrightarrow{h} & D & \xrightarrow{t} & C \\ & \searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & A & \xrightarrow[s]{} & B \end{array}$$

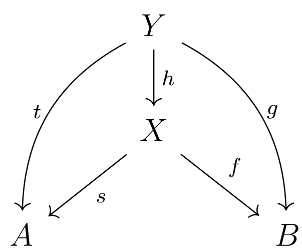
In the diagram above, $f \in \mathrm{Mor}(\mathcal{D}), s \in S \cap \mathrm{Mor}(\mathcal{D})$. Then at the level of $\mathcal{C}$, there are morphisms t, g such that $f \circ t = s \circ g$, with $t \in S$. According to the assumption there will exist a morphism $h : E \to D$ such that $E \in \mathrm{Ob}(\mathcal{D}), t \circ h \in S \cap \mathrm{Mor}(\mathcal{D})$. This proves the condition of right Ore.

We assume $f, g \in \mathrm{Mor}(\mathcal{D})$ and $s \in S \cap \mathrm{Mor}(\mathcal{D})$ such that $s \circ f = s \circ g$. Then we have the following diagram:

$$E \xrightarrow{h} D \xrightarrow{t} A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{s} C$$

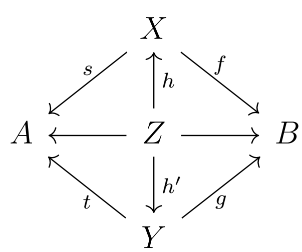
where $t \in S$ but $t \circ h \in S \cap \mathrm{Mor}(\mathcal{D})$, $f \circ t = g \circ t$. This proves the right reversible condition. Next we should prove $\mathcal{D}$ satisfies two conditions of localizing subcategory.

1. Given a morphism $(f, s) : A \to B$ in $\mathcal{C}S^{-1}$ with $A, B \in \mathrm{Ob}(\mathcal{D})$, there will exist $h : Y \to X$ with $Y \in \mathrm{Ob}(\mathcal{D})$ such that $s \circ h \in S \cap \mathrm{Mor}(\mathcal{D})$.



Then $(f, s) \sim (g, t)$ with $(f, s) \circ h = (g, t)$.

2. Given morphisms $(f, s) \sim (g, t)$ identified in $\mathcal{C}S^{-1}$ with $f, g \in \mathrm{Mor}(\mathcal{D})$ and $s, t \in S \cap \mathrm{Mor}(\mathcal{D})$, there are morphisms $h, h' \in \mathrm{Mor}(\mathcal{C})$ such that $(fh, sh) = (gh', th')$.



Though $Z \in Ob(\mathcal{C})$, we can find $E \to Z$ with $E \in \mathrm{Ob}(\mathcal{D})$ and $E \to Z \to A \in S \cap \mathrm{Mor}(\mathcal{D})$ since $Z \to A \in S$ and $A \in Ob(\mathcal{D})$. Then (f, s) and (g, t) are identified in $\mathcal{D}S^{-1}$. $\square$

Dually there is a similar theorem for calculus of left fractions. We assume $(\mathcal{C}, S)$ admits a calculus of left fractions and $\mathcal{D}$ is a full subcategory of it such that $\forall s \in X \to Y \in S, X \in \mathrm{Ob}(\mathcal{D})$, $\exists f : Y \to Z$ with $Z \in \mathrm{Ob}(\mathcal{D}), f \circ s : X \to Y \to Z \in S \cap \mathrm{Mor}(\mathcal{D})$. Then $(\mathcal{D}, S \cap \mathrm{Mor}(\mathcal{D}))$ is a localizing subcategory admitting a calculus of left fractions.

We will use this theorem later to obtain bounded derived categories.

3.4 Triangulated Categories

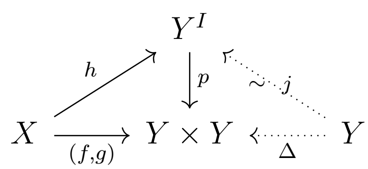
We assume $\mathcal{A}$ is an abelian category and $\mathbf{Ch}(\mathcal{A})$ is the category of chain complexes. $\mathbf{K}(\mathcal{A})$ is the quotient category of $\mathbf{Ch}(\mathcal{A})$, which means $\mathbf{K}(\mathcal{A})$ has the same objects as $\mathbf{Ch}(\mathcal{A})$ and $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(X, Y)/\sim$, where $f \sim g$ if f is chain homotopic to g.

If the class of quasi-isomorphisms (homology isomorphisms) is denoted by $\mathcal{W}$, then the derived category of $\mathcal{A}$ is just $\mathbf{Ch}(\mathcal{A})[\mathcal{W}^{-1}]$ which is written as $\mathbf{D}(\mathcal{A})$ as well. The previous chapter tells us $\mathbf{Ch}(\mathcal{A})$ is actually a model category and we can use homotopies between fibrant and cofibrant objects to describe morphisms in $\mathbf{Ch}(\mathcal{A})[\mathcal{W}^{-1}]$. In this way we can deal with $\mathbf{Ch}(\mathcal{A})$ directly.

Before Quillen, there is another method to describe morphisms in $\mathbf{Ch}(\mathcal{A})[\mathcal{W}^{-1}]$, which is due to Verdier. Instead of dealing with $\mathbf{Ch}(\mathcal{A})$ directly, we deal with $\mathbf{K}(\mathcal{A})$ and prove $(\mathbf{K}(\mathcal{A}), \mathcal{W})$ admits a calculus of left and right fractions. The reason why we can deal with $\mathbf{K}(\mathcal{A})$ instead of $\mathbf{Ch}(\mathcal{A})$ is explained in the Corollary 3.21. To prove the existence of calculus of fractions, we need the technique from triangulated categories. In this section, we develop some general theories of triangulated categories and in the next section we prove $\mathbf{K}(\mathcal{A})$ is a triangulated category and $(\mathbf{K}(\mathcal{A}), \mathcal{W})$ admits a calculus of left and right fractions.

Theorem 3.20. $\mathbf{Ch}(\mathcal{A})[\mathcal{H}^{-1}] \cong \mathbf{K}(\mathcal{A})$, *where $\mathcal{H}$ is the class of chain homotopy equivalences.*

Proof. We only need to prove for every functor $F : \mathbf{Ch}_{\cdot}(\mathcal{A}) \to \mathcal{D}$, if F takes chain homotopy equivalences to isomorphisms, then $F(f) = F(g)$ whenever f is chain homotopic to g. The Example 2.34 implies chain homotopies are the same as right homotopies. Therefore, if f is chain homotopic to g, $f - g = s\partial + \partial s$, then we have the following diagram:



where $Y_n^I \cong Y_n \oplus Y_n \oplus Y_{n+1}$, $h = (f, g, (-1)^{n+1}s)$, $j = (\mathrm{id}_Y, \mathrm{id}_Y, 0)$. Now we only need to prove j is a chain homotopy equivalence. If it's true, $F(j)$ will be an isomorphism, then $F(f) = F(g) = F(j)^{-1}F(h)$. Define $q : Y^I \to Y$, $q(x, y, z) = x$. Then $q \circ j = \mathrm{id}_Y$. $j \circ q(x, y, z) = (x, x, 0)$, $(\mathrm{id} - j \circ q)(x, y, z) = (0, y - x, z)$.

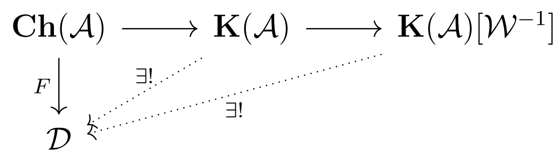
We let $s'_n : Y_n \oplus Y_n \oplus Y_{n+1} \to Y_{n+1} \oplus Y_{n+1} \oplus Y_{n+2}$, $(x, y, z) \mapsto \big(0, (-1)^n z, 0\big)$.

$$\begin{aligned}
&(\partial s'_n + s'_{n-1}\partial)(x, y, z)\\
=&\partial\big(0, (-1)^n z, 0\big) + s'_{n-1}\big(\partial x, \partial y, (-1)^n(x - y) + \partial z\big)\\
=&\big(0, (-1)^n \partial z, -(-1)^{n+1}(-1)^n z\big) + \big(0, (-1)^{n-1}[(-1)^n(x - y) + \partial z], 0\big)\\
=&\big(0, (-1)^n \partial z, z\big) + \big(0, y - x + (-1)^{n-1}\partial z, 0\big)\\
=&(0, y - x, z)
\end{aligned}$$

This means j is a chain homotopy equivalence. $\square$

Corollary 3.21. *Since every chain homotopy equivalence is a quasi-isomorphism, we denote the class of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ by $\mathcal{W}$ as well. Then $\mathbf{Ch}(\mathcal{A})[\mathcal{W}^{-1}] \cong \mathbf{K}(\mathcal{A})[\mathcal{W}^{-1}]$.*

Proof.



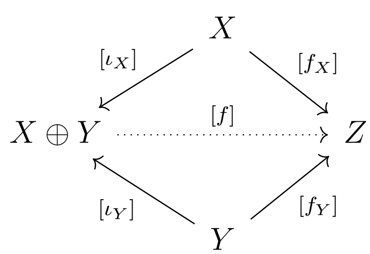
where F is a functor taking quasi-isomorphisms to isomorphisms. □

Therefore we can just deal with $(\mathbf{K}(\mathcal{A}), \mathcal{W})$. Although $\mathbf{K}(\mathcal{A})$ is additive (which is not so obvious, see Proposition 3.22), in general it's not abelian, which will be clear if we know $\mathbf{K}(\mathcal{A})$ is a triangulated category. A category which is both abelian and triangulated is very special. In fact, the category of this kind is *semi-simple*, which means every short exact sequence in it splits. This will be proved later. In the following we will prove $(\mathbf{K}(\mathcal{A}), \mathcal{W})$ admits a calculus of left and right fractions which is actually an easy corollary of the fact that $\mathbf{K}(\mathcal{A})$ is a triangulated category.

Proposition 3.22. $\mathbf{K}(\mathcal{A})$ *is an additive category.*

Proof. $[f] + [g] = [f + g]$ is well defined since if $f \sim f'$ and $g \sim g'$, then $f - f' = \partial s + s\partial$, $g - g' = \partial s' + s'\partial$ and $(f + g) - (f' + g') = \partial(s + s') + (s + s')\partial$. 0 is also the zero object in $\mathbf{K}(\mathcal{A})$. Now we only need to prove $X \oplus Y$ is the coproduct in $\mathbf{K}(\mathcal{A})$ as well.

Given a coproduct diagram



f exists at the level of $\mathbf{Ch}(\mathcal{A})$ and we prove it's unique up to chain homotopy. If there is some $[g] : X \oplus Y \to Z$ satisfying $[g\iota_X] = [f_X]$ and $[g\iota_Y] = [f_Y]$. Then $g \circ \iota_X \sim f \circ \iota_X$, $g_n\iota_{X_n} - f_n\iota_{X_n} = \partial_{n+1}s_n^X + s_{n-1}^X\partial_n$ where $s_n^X : X_n \to Z_{n+1}$. Let

$$s_n = (s_n^X, s_n^Y) : X_n \oplus Y_n \to Z_n$$

Then $g - f = \partial s + s\partial$ which can be checked using the universal property of coproducts in $\mathbf{Ch}(\mathcal{A})$. □

Now let us introduce the concept of triangulated categories. Assume $\mathcal{A}$ is an additive category and $T : \mathcal{A} \to \mathcal{A}$ is called the *translation functor* if it's an additive automorphism which means T^{-1} exists and satisfies $T \circ T^{-1} = \mathrm{id}_{\mathcal{A}}$, $T^{-1} \circ T = \mathrm{id}_{\mathcal{A}}$. A *triangle* in $\mathcal{A}$ is a sequence

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX$$

and the morphism between triangles is just a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

Definition 3.23. *A **pre-triangulated** category is an additive category $\mathcal{A}$ together with a translation functor T and a collection of distinguished triangles satisfying following axioms*

(TR0) Any triangle isomorphic to a distinguished one is distinguished as well.

(TR1) For any object $X \in \mathrm{Ob}(\mathcal{A})$, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow TX$$

is distinguished.

(TR2) For any morphism $f : X \to Y$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

(TR3)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is distinguished iff

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$

is distinguished.

(TR4) Given distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\vdots\,{\scriptstyle \exists} & & {\scriptstyle Tf}\downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' \end{array}$$

such that $u'f = gu$ there will exist a map $h : Z \to Z'$ making the diagram above commutative.

We call such category *pre-triangulated* since for a true *triangulated category* it should satisfy one more axiom (TR5) (see Definition 3.34) which is much more complicated than others we have described above. Actually we can obtain many properties just from axioms above and do not use axiom (TR5).

Fact 3.24. From (TR1) and (TR3) we know

$$X \longrightarrow 0 \longrightarrow TX \xrightarrow{-\mathrm{id}_{TX}} TX$$

$$0 \longrightarrow TX \xrightarrow{-id} TX \longrightarrow 0$$

$$TX \xrightarrow{-\mathrm{id}} TX \longrightarrow 0 \longrightarrow T^2X$$

$$0 \longrightarrow T^2X \xrightarrow{\mathrm{id}} T^2X \longrightarrow 0$$

are all distinguished. Since X is arbitrary, we can choose $X = T^{-1}Y$ or $X = T^{-2}(Y)$ and we conclude

$$0 \longrightarrow X \xrightarrow{\pm \mathrm{id}} X \longrightarrow 0$$

is distinguished.

Only from (TR3) we see

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{W} TX$$

is distinguished iff

$$TX \xrightarrow{-Tu} TY \xrightarrow{-Tv} TZ \xrightarrow{-Tw} T^2X$$

is distinguished.

Fact 3.25. In a pre-triangulated category $(\mathcal{A}, T)$ there are other two equivalent forms of (TR4).

One is

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\,\vdots\,{\scriptstyle \exists} & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' \end{array}$$

where it should satisfy $w' \circ h = Tf \circ w$.

The othere is

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle f}\,\vdots\,{\scriptstyle \exists} & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\,\vdots \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' \end{array}$$

where it should satisfy $v'g = hv$.

This fact can be proved only using (TR3) to change positions of maps $f,\ g,\ h$.

Fact 3.26. *If $(\mathcal{A}, T)$ is a pre-triangulated category and*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is distinguished, then $v \circ u = 0$ and $w \circ v = 0$

Proof. From (TR3)

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$

is distinguished. Hence we only need to prove $v \circ u = 0$.

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY \\ {\scriptstyle v}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \vdots & & \downarrow \\ Z & \xrightarrow[\mathrm{id}]{} & Z & \longrightarrow & 0 & \longrightarrow & TZ \end{array}$$

Then $Tv \circ (-Tu) = -T(vu) = 0$ which means $vu = 0$. $\square$

Definition 3.27. *If $(\mathcal{A}, T)$ is a pre-triangulated category and $\mathcal{B}$ is an abelian category, the functor $H : \mathcal{A} \to \mathcal{B}$ is called* ***homological*** *if for any distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

the following sequence in $\mathcal{B}$ is exact

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

There is a dual concept for *cohomological functors* which are functors $H : \mathcal{A}^{op} \to \mathcal{B}$ making

$$H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X)$$

exact.

Remark 3.28. From (TR3)

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$
$$T^{-1}Z \xrightarrow{-T^{-1}w} X \xrightarrow{u} Y \xrightarrow{v} Z$$

are distinguished. And note that for a morphism f in an abelian category its kernel and image coincide with that of $-f$ respectively. Therefore we obtain the following long exact sequence of abelian groups via a homological functor H.

$$\cdots \longrightarrow H(T^{-1}Z) \xrightarrow{H(T^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(TX) \longrightarrow \cdots$$

Theorem 3.29. *Suppose $(\mathcal{A}, T)$ is a pre-triangulated category. Then the representable functor $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathbf{Ab}$ is homological for any $A \in \mathrm{Ob}(\mathcal{A})$. Dually $\mathrm{Hom}(-, A)$ is cohomological.*

Proof. Given a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

We only need to prove

$$\mathrm{Hom}(A, X) \xrightarrow{u_*} \mathrm{Hom}(A, Y) \xrightarrow{v_*} \mathrm{Hom}(A, Z)$$

is exact. Since $v \circ u = 0$, $v_* \circ u_* = 0$. If $f \in \ker v_*$, then $v_*(f) = v \circ f = 0$.

$$\begin{array}{ccccccc}
A & \longrightarrow & 0 & \longrightarrow & TA & \xrightarrow{-\mathrm{id}} & TA \\
{\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle Tg}\,\vdots\,{\scriptstyle \exists} & & {\scriptstyle Tf}\downarrow \\
Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX & \xrightarrow[-Tu]{} & TY
\end{array}$$

Then $-Tu \circ Tg = Tf \circ (-\mathrm{id})$ Then $T(ug) = Tf \Rightarrow ug = f$. $\square$

Theorem 3.30 (Triangulated 5-Lemma)**.** *Given a morphism between two distinguished triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TA \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' \end{array}$$

if f, g are isomorphisms, then so is h. In fact any two of f, g, h are isomorphisms, so is the remaider.

Proof. The second statement is clear by (TR3). For any $A \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{array}{ccccccccc} \mathrm{Hom}(A,X) & \xrightarrow{u_*} & \mathrm{Hom}(A,Y) & \xrightarrow{v_*} & \mathrm{Hom}(A,Z) & \xrightarrow{w_*} & \mathrm{Hom}(A,TX) & \xrightarrow{T(u)_*} & \mathrm{Hom}(A,TY) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle g_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle h_*}\downarrow & & {\scriptstyle Tf_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle Tg_*}\downarrow{\scriptstyle \cong} \\ \mathrm{Hom}(A,X') & \xrightarrow[u'_*]{} & \mathrm{Hom}(A,Y') & \xrightarrow[v'_*]{} & \mathrm{Hom}(A,Z') & \xrightarrow[w'_*]{} & \mathrm{Hom}(A,TX') & \xrightarrow[T(u')_*]{} & \mathrm{Hom}(A,TY') \end{array}$$

Then according to the 5-lemma in $\mathbf{Ab}$, h_* is an isomorphism. By Yoneda's lemma h will be an isomorphism. □

Now we want to compare two notions of abelian categories and pre-triangulated categories.

Definition 3.31. *An abelian category $\mathcal{A}$ is **semi-simple** if every short exact sequence in $\mathcal{A}$ splits.*

Lemma 3.32. *In a pre-triangulated category $\mathcal{A}, T$, if*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{0} TX$$

is distinguished, then this triangle splits, i.e. u is a split monomorphism and v is a split epimorphism.

Proof.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & Z & \longrightarrow & TX \\ {\scriptstyle \mathrm{id}}\downarrow & & \vdots{\scriptstyle u'} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow[\mathrm{id}]{} & X & \longrightarrow & 0 & \longrightarrow & TX \end{array}$$

According to Fact 3.25, the dotted arrow u' exists. Then $u'u = \mathrm{id}$.

$$\begin{array}{ccccccc} Z & \xrightarrow{\mathrm{id}} & Z & \longrightarrow & 0 & \longrightarrow & TZ \\ {\scriptstyle v'}\vdots & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \vdots{\scriptstyle Tv'} \\ Y & \xrightarrow[v]{} & Z & \xrightarrow[0]{} & TX & \xrightarrow[-Tu]{} & TY \end{array}$$

This diagram implies $vv' = \mathrm{id}$. □

Theorem 3.33. *If $\mathcal{A}$ is both abelian and pre-triangulateed, then it's semi-simple.*

Proof. If

$$0 \longrightarrow X \overset{f}{\to} Y \overset{g}{\to} Z \longrightarrow 0$$

is exact in $\mathcal{A}$. For the morphism f, by (TR2) there will exist a distinguished triangle

$$X \overset{f}{\to} Y \overset{u}{\to} W \overset{v}{\to} TX$$

Then by (TR3)

$$T^{-1}W \xrightarrow{-T^{-1}v} X \overset{f}{\to} Y \overset{u}{\to} W$$

is distinguished which imiplies $f \circ T{-}1v = 0$. But f is monic and this means $T^{-1}v = 0 \Rightarrow v = 0$. Then

$$X \overset{f}{\to} Y \overset{u}{\to} W \overset{0}{\to} TX$$

splits by the previous Lemma 3.32. Then f admits a retraction $f' : Y \to X$ which make the original exact seqence split. □

Above we always talk about pre-triangulated categories. Now let us introduce the real definition of triangulated categories which will be used in the next section.

Definition 3.34. *A **triangulated** category is a pre-triangulated category $(\mathcal{A}), T)$ satisfying the following axiom*

(TR5) (Octahedra Axiom) Given three distinguished triangles

$$X \overset{u}{\to} Y \longrightarrow Z' \longrightarrow TX$$
$$Y \overset{v}{\to} Z \longrightarrow X' \longrightarrow TY$$
$$X \xrightarrow{v \circ u} Z \longrightarrow Y' \longrightarrow TX$$

there is a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow TZ'$$

making the following diagram commutative

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & TX \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{v \circ u} & Z & \longrightarrow & Y' & \longrightarrow & TX \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}} & & \vdots & & \downarrow{\scriptstyle Tu} \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & TY \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
Z' & \cdots\rightarrow & Y' & \cdots\rightarrow & X' & \longrightarrow & TZ'
\end{array}$$

Since this axiom is really complicated, we want to state some motivations for it. In fact given morphisms $X \overset{u}{\to} Y \overset{v}{\to} Z$, X', Y', Z' can be obtained from (TR2) and dotted morphisms $Z' \to Y'$, $Y' \to X'$ can be obtained by (TR4). (TR5) actually says among all such choices there is the "best" one or the most "functorial" one. However, in spite of the existence of this "best" choice, the *cone construction* in a triangulated category is still not functorial, which is the most serious flaw for the concept of triangulated categories and there are many attempts to fix it such as Grothendieck's *derivator*. In some sense the theory of *infinite category* is also a such attempt, since the homotopy theory of a *stable model category* is triangulated and there is a generalization for this special type of triangulated categories named *stable infinite category* developed by Lurie. Nowadays, there is also another satisfactory theory to fix this flaw. In the definition of triangulated categories, we can replace the additive category by a *differential graded category* and we will obtain the *dg-triangulated category* which has functorial cone constructions in the derived or homotopical sense.

Roughly speaking, in an abelian sense (TR5) says given abelian groups $A \subseteq B \subseteq C$ and exact sequences $A \to B \to B/A$, $A \to C \to C/A$, $B \to C \to C/B$,

$$B/A \to C/A \to C/B$$

should be exact. This is a good explanation at the level of abelian groups. There are also some equivalent versions of (TR5) and readers interested in it can consult [Nee01].

3.5 The Derived Category $\mathbf{D}(\mathcal{A})$

For a ring R (resp. an abelian category $\mathcal{A}$) its *derived category* is defined to be the localization of $\mathbf{Ch}(R)$ (resp. $\mathbf{Ch}(\mathcal{A})$) with respect to quasi-isomorphisms. In this section our task is to prove $\mathbf{K}(R)$ (resp. $\mathbf{K}(\mathcal{A})$) admits a calculus of left and right fractions and we will use this technique to construct $\mathbf{D}(R) := \mathbf{K}(R)[\mathcal{W}^{-1}]$ (resp. $\mathbf{D}(\mathcal{A}) := \mathbf{K}(\mathcal{A})[\mathcal{W}^{-1}]$).

We define the translation functor as follows.

$$T : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A}),\ X \mapsto X[1]$$

where $X[1]_n = X_{n-1}$ and $\partial_n^{X[1]} = -\partial_{n-1}^X$. For any chain map f, $Tf = f[1]$ where $f[1]_n = f_{n-1}$. $T^{-1} = [-1]$. Obviously T is compatible with chain homotopies and then $T : \mathbf{K}(\mathcal{A}) \xrightarrow{\sim} \mathbf{K}(\mathcal{A})$.

Definition 3.35. *For a chain map $f : X \to Y$, its **mapping cone** $M(f)$ is defined to be*

$$M(f)_n = X_{n-1} \oplus Y_n = X[1]_n \oplus Y_n$$

and its differential is

$$\partial_n : X_{n-1} \oplus Y_n \to X_{n-2} \oplus Y_{n-1},\ (x, y) \mapsto (-\partial_{n-1}^X(x), f_{n-1}(x) + \partial_{n-1}^Y(y))$$

You can compare this concept with that in algebraic topology.

Fact 3.36. $M(f)$ is actually a complex.

$$\begin{aligned}\partial\partial(x, y) &= \partial(-\partial x, f(x) + \partial y) \\ &= (\partial\partial x, -f(\partial x) + \partial f(x) + \partial\partial y) \\ &= 0\end{aligned}$$

Remark 3.37. For any chain map f, we will have

$$\alpha_f : Y \to M(f),\ y \mapsto (0, y)$$

It's a chain map since $\partial\alpha_f(y) = \partial(0, y) = (0, \partial y) = \alpha_f(\partial y)$.

And we can define another chain map

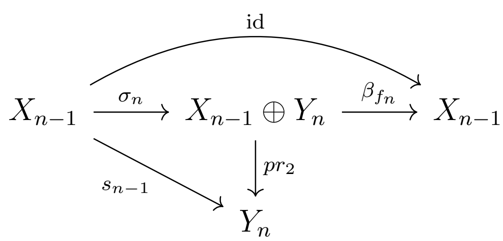
$$\beta_f : M(f) \to X[1],\ (x, y) \mapsto x$$

Then $\beta_f \partial(x, y) = -\partial x = \partial^{X[1]} x = -\partial x$. Finally we obtain the short exact sequence

$$0 \longrightarrow Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1] \longrightarrow 0$$

Lemma 3.38. *The sequence above in Remark 3.37 splits iff f is homotopic to 0.*

Proof. "$\Rightarrow$". Suppose β_f admits a section $\sigma : X[1] \to M(f)$ such that $\beta_f \circ \sigma = \mathrm{id}_{X[1]}$.



We define $s_{n-1} = pr_2 \circ \sigma_n$, which means $\sigma_n(x) = (x, s_{n-1}(x))$. Since σ is a chain map

$$\begin{aligned} \partial\sigma(x) &= \partial(x, sx) = (-\partial x, fx + \partial sx) \\ &= \sigma\partial^{X[1]} x = \sigma(-\partial x) = (-\partial x, -s\partial x) \end{aligned}$$

Then $fx + \partial sx = -s\partial x$, which means $f \sim 0$.

"$\Leftarrow$". If $f \sim 0$, then there will exist $s_{n-1} : X_{n-1} \to Y_n$ such that $s\partial + \partial s = -f$. Define

$$\sigma : X[1] \to M(f),\ x \mapsto (x, sx)$$

According to the part "$\Rightarrow$" we see it's actually a chain map and $\beta_f \circ \sigma = \mathrm{id}_{X[1]}$. □

Example 3.39.

(1) Define $X = A(0)$ which means the only non-zero object in this chain complex is A at the position $n = 0$. Let $Y = B(0)$. Given any morphism $f : A \to B$, then

$$M(f) = \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \longrightarrow \cdots$$

where B is at the position $n = 0$ and A is at $n = 1$.

(2) For the identity map $\mathrm{id} : X \to X$, $M(\mathrm{id}) = X_{n-1} \oplus X_n$. We could define

$$s_n : X_{n-1} \oplus X_n \mapsto X_n \oplus X_{n+1},\ (x_{n-1}, x_n) \mapsto (x_n, 0)$$

then

$$
\begin{aligned}
&(\partial s + s\partial)(x_{n-1}, x_n) \\
=&\partial(x_n, 0) + s(\partial x_{n-1}, x_{n-1} + \partial x_n) \\
=&(-\partial x_n, x_n) + (x_{n-1} + \partial x_n, 0) \\
=&(x_{n-1}, x_n)
\end{aligned}
$$

This means $\mathrm{id}_{M(\mathrm{id})} \sim 0$. Hence in $\mathbf{K}(\mathcal{A})$, $M(\mathrm{id})$ is isomorphic to 0.

Proposition 3.40. *Let $f : X \to Y$ be chain map. Then f is an quasi-isomorphism iff $M(f)$ is exact which means all its homology groups are trivial.*

Proof. From Remark 3.37, we see

$$0 \longrightarrow Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1] \longrightarrow 0$$

is exact. Then we have the long exact sequence

$$\cdots \to H_n(Y) \to H_n(M(f)) \to H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \to \cdots$$

To see why the connecting map $H_{n-1}(X) \to H_{n-1}(Y)$ is just f_*, we compute is concretely.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y_n & \longrightarrow & X_{n-1} \oplus Y_n & \overset{\dashleftarrow}{\longrightarrow} & X_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \vdots\downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_{n-1} & \overset{\dashleftarrow}{\longrightarrow} & X_{n-2} \oplus Y_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & 0
\end{array}
$$

The connecting morphism is just the dotted arrows

$$x \mapsto (x, 0) \mapsto (-\partial x, fx) \mapsto fx$$

□

Now let us define the triangulated category structure on $\mathbf{K}(\mathcal{A})$.

Definition 3.41. *In $\mathbf{K}(\mathcal{A})$ a sequence having the following form*

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1]$$

is called a standard triangle. Any triangle isomorphic to a standard one is distinguished.

Theorem 3.42. *Let $\mathcal{A}$ be an abelian category. Then $\mathbf{K}(\mathcal{A})$ is a triangulated category.*

Proof. Actually $\mathcal{A}$ can be any additive category and to prove this theorem we won't need kernels and cokernels.

At first we prove in $\mathbf{Ch}(\mathcal{A})$, $\alpha_f \circ f$ is homotopic to 0 and $\beta_f \circ f = 0$. The second one is clear and hence we only need to prove the first one.

$$\alpha_f \circ f : X \to M(f),\ x_n \mapsto (0, f(x_n))$$

Define

$$s_n : X_n \to X_n \oplus Y_{n+1},\ x_n \mapsto (x_n, 0)$$

then

$$\begin{aligned}
&\partial s_n(x_n) + s_{n-1}\partial(x_n)\\
=&\partial(x_n, 0) + (\partial x_n, 0)\\
=&(-\partial x_n, f x_n) + (\partial x_n, 0)\\
=&(0, f x_n) = \alpha_f \circ f(x_n)
\end{aligned}$$

(TR0) and (TR1) hold by definition.

(TR1). Since

$$X \xrightarrow{\text{id}} X \longrightarrow M(\text{id}) \longrightarrow X[1]$$

is distinguished and $M(\text{id})$ is isomorphic to 0 by Example 3.39 (2), we have following isomorphic triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & X[1]\\
\Big\downarrow{\scriptstyle \text{id}} & & \Big\downarrow{\scriptstyle \text{id}} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow\\
X & \xrightarrow[\text{id}]{} & X & \longrightarrow & M(\text{id}) & \longrightarrow & X[1]
\end{array}$$

(TR3). For a standard triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1]$$

we only need to prove

$$Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1] \xrightarrow{-f} Y[1]$$

is distinguished. If this is valid, and

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished, then we see

$$X[1] \xrightarrow{-u[1]} Y[1] \xrightarrow{-v[1]} Z[1] \xrightarrow{-w[1]} X[2]$$

is distinguished and isomorphic to a standard triangle which will imply

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is isomorphic to a standard one.

At first we have

$$M(\alpha_f)_n = Y_{n-1} \oplus X_{n-1} \oplus Y_n,\ (y_{n-1}, x_{n-1}, y_n) \mapsto (-\partial y_{n-1}, -\partial x_{n-1}, y_{n-1} + f x_{n-1} + \partial y_n)$$

We want to prove there is an isomorphism g making the following diagram commutative

$$\begin{array}{ccccccc} Y & \xrightarrow{\alpha_f} & M(f) & \xrightarrow{\beta_f} & X[1] & \xrightarrow{-f} & Y[1] \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle g}\vdots{\scriptstyle \exists} & & {\scriptstyle \mathrm{id}}\downarrow \\ Y & \xrightarrow[\alpha_f]{} & M(f) & \xrightarrow[\alpha_{\alpha_f}]{} & M(\alpha_f) & \xrightarrow[\beta_{\alpha_f}]{} & Y[1] \end{array}$$

We write it concretely

$$\begin{array}{ccccccc} Y_n & \xrightarrow{\alpha_f} & X_{n-1}\oplus Y_n & \xrightarrow{\beta_f} & X_{n-1} & \xrightarrow{-f} & Y_{n-1} \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle j} & & {\scriptstyle \mathrm{id}}\downarrow \\ Y_n & \xrightarrow[\alpha_f]{} & X_{n-1}\oplus Y_n & \xrightarrow[\alpha_{\alpha_f}]{} & Y_{n-1}\oplus X_{n-1}\oplus Y_n & \xrightarrow[\beta_{\alpha_f}]{} & Y_{n-1} \end{array}$$

Define $g(x)=(-fx,x,0)$. Firstly we check it's a chain map.

$$\partial g(x)=\partial(-fx,x,0)=(\partial fx,-\partial x,-fx+fx)=(\partial fx,-\partial x)$$

and

$$g\partial^{X[1]}(x)=-g\partial x=(f\partial x,-\partial x,0)$$

Next we prove such defined g make the diagram commutative up to chain homotopy, which is what the difficulty lies in. In the long proof of this theorem, "commutative diagram" is always commutative up to homotopy and we should find all chain homotopies. Consider $\alpha_{\alpha_f}-g\circ\beta_f$ which sends (x,y) to $(fx,0,y)$.

Define

$$s_n: X_{n-1}\oplus Y_n\to Y_n\oplus X_n\oplus Y_{n+1},\ (x,y)\mapsto(y,0,0)$$

then

$$\begin{aligned}(\partial s+s\partial)(x,y)&=\partial(y,0,0)+s(-\partial x,fx+\partial y)\\&=(-\partial y,0,y)+(fx+\partial y,0,0)\\&=(fx,0,y)\end{aligned}$$

Then in $\mathbf{K}(\mathcal{A})$, $\alpha_{\alpha_f}=g\circ\beta_f$. And note that $\beta_{\alpha_f}\circ g=-f$ is strict. Next we will prove g is a homotopy equivalence.

Let $j: M(\alpha_f)\to X[1],\ Y_{n+1}\oplus X_{n-1}\oplus Y_n\to X_{n-1}$ by $(y_1,x,y_2)\mapsto x$. Also we should prove j is actually a chain map

$$j\partial(y_1,x,y_2)=j(-\partial y_1,-\partial x,y_1+fx+\partial y+2)=-\partial x$$

and $\partial^{X[1]}j(y_1,x,y_2)=\partial^{X[1]}x=-\partial x$. Obviously we see $j\circ g=\mathrm{id}_X$. Hence we only need to prove $g\circ j\sim\mathrm{id}$. Actually $g\circ j(y_1,x,y_2)=(-fx,x,0)$ then

$$(gj-\mathrm{id})(y_1,x,y_2)=(y_1+fx,0,y_2)$$

Let $s'_n : Y_{n-1} \oplus X_{n-1} \oplus Y_n \to Y_n \oplus X_n \oplus Y_{n+1},\ (y_1, x, y_2) \mapsto (y_2, 0, 0)$. Then

$$\begin{aligned}(\partial s' + s'\partial)(y_1, x, y_2) &= \partial(y_2, 0, 0) + s'(-\partial y_1, -\partial x, y_1 + fx + \partial y_2) \\ &= (-\partial y_2, 0, y_2) + (y_1 + fx + \partial y_2, 0, 0) \\ &= (y_1 + fx, 0, y_2)\end{aligned}$$

These prove (TR3).

(TR4). We only need to find the morphism w between two standard triangles.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha_f} & M(f) & \xrightarrow{\beta_f} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\,\vdots\,{\scriptstyle \exists} & & {\scriptstyle u[1]}\downarrow \\ A & \xrightarrow[g]{} & B & \xrightarrow[\alpha_g]{} & M(g) & \xrightarrow[\beta_g]{} & A[1] \end{array}$$

Also we write it concretely

$$\begin{array}{ccccccc} X_n & \xrightarrow{f} & Y_n & \xrightarrow{\alpha_f} & X_{n-1} \oplus Y_n & \xrightarrow{\beta_f} & X_{n-1} \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\,\vdots & & {\scriptstyle u}\downarrow \\ A_n & \xrightarrow[g]{} & B_n & \xrightarrow[\alpha_g]{} & A_{n-1} \oplus B_n & \xrightarrow[\beta_g]{} & A_{n-1} \end{array}$$

Since $gu \sim vf$, there will exist $s_n : X_n \to B_{n+1}$ such that $vf - gu = s\partial + \partial s$. We let $w(x, y) = (ux, sx + vy)$. Then obviously we see in $\mathbf{Ch}(\mathcal{A})$, $\alpha_g v = w\alpha_f$ and $\beta_g w = u\beta_f$. Therefore we only need to check it's actually a chain map.

$$w\partial(x, y) = w(-\partial x, fx + \partial y) = (-u\partial x, -s\partial x + vfx + v\partial y)$$

and $\partial w(x, y) = \partial(ux, sx + vy) = (-\partial ux, gux + \partial sx + \partial vy)$. They in fact coincide with each other.

(TR5). This is the most complicated part of the proof and it can also be checked at the level of standard triangles. Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha_f} & M(f) & \xrightarrow{\beta_f} & X[1] \\ \downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle u}\,\vdots & & \downarrow \\ X & \xrightarrow{gf} & Z & \xrightarrow{\alpha_{gf}} & M(gf) & \xrightarrow{\beta_{gf}} & X[1] \\ {\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle v}\,\vdots & & \downarrow \\ Y & \xrightarrow[g]{} & Z & \xrightarrow[\alpha_g]{} & M(g) & \xrightarrow[\beta_g]{} & Y[1] \end{array}$$

If we let $u = (\mathrm{id}, g) : X[1] \oplus Y \to X[1] \oplus Z$ and $v = (f, \mathrm{id}) : X[1] \oplus Z \to Y[1] \oplus Z$, then the right part of this diagram will be commutative in $\mathbf{Ch}(\mathcal{A})$. It's easy to see u and v are chain maps. Next our task is to prove

$$M(f) \xrightarrow{u} M(gf) \xrightarrow{\mu} M(f)[1]$$

where $\mu(y,z) = (0,y)$ is distinguished. Specifically, we prove it is isomorphic to

$$M(f) \xrightarrow{u} M(gf) \xrightarrow{\alpha_u} M(u) \xrightarrow{\beta_u} M(f)[1]$$

up to homotopy.

$$\begin{array}{ccccccc}
X_{n-1} \oplus Y_n & \xrightarrow{u} & X_{n-1} \oplus Z_n & \xrightarrow{v} & Y_{n-1} \oplus Z_n & \xrightarrow{\mu} & X_{n-2} \oplus Y_{n-1} \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow w & & \downarrow \mathrm{id} \\
X_{n-1} \oplus Y_n & \xrightarrow[u]{} & X_{n-1} \oplus Z_n & \xrightarrow[\alpha_u]{} & X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n & \xrightarrow[\beta_u]{} & X_{n-2} \oplus Y_{n-1}
\end{array}$$

where $M(u)_n := M(f)_{n-1} \oplus M(gf)_n = X_{n-2} \oplus Y_n \oplus X_{n-1} \oplus Z_n$ and the differential operator is actually

$$\begin{aligned}
\partial(x_1, y, x_2, z) &= \big(-\partial(x,y), u(x,y) + \partial(x_2, z) \big) \\
&= \big(-(\partial x_1, fx_1 + \partial y), (x_1, gy) + (-\partial x_2, gfx_2 + \partial z) \big) \\
&= (\partial x_1, -fx_1 - \partial y, x_1 - \partial x_2, gfx_2 + gy + \partial z)
\end{aligned}$$

We define $w(y,z) = (0, y, 0, z)$, then

$$w\partial(y,z) = w(-\partial y, gy + \partial z) = (0, -\partial y, 0, gy + \partial z)$$

and $\partial w(y,z) = \partial(0,y,0,z) = (0, -\partial y, 0, gy + \partial z)$. Thus w is a chain map.

$\beta_u \circ w(y,z) = (0,y) = \mu(y,z)$. Then $\beta_u \circ w = \mu$ in $\mathbf{Ch}(\mathcal{A})$.

$$wv(x,z) = w(fx, z) = (0, fx, 0, z), \quad \alpha_u(x,z) = (0, 00, x, z)$$

hence $(\alpha_u - wv)(x,z) = (0, -fx, x, 0)$. Define

$$s_n : X_{n-1} \oplus Z_n \to X_{n-1} \oplus Y_n \oplus X_n \oplus Z_{n+1},\ (x,z) \mapsto (x, 0, 0, 0)$$

and then

$$\begin{aligned}
\partial s + s\partial)(x,z) &= \partial(x,0,0,0) + s(-\partial x, gfx + \partial z) \\
&= (\partial x, -fx, x, 0) + (-\partial x, 0, 0, 0) \\
&= (0, -fx, x, 0)
\end{aligned}$$

which means $\alpha_u = wv$ up to homotopy. This proves the diagram above is commutative. Finally we want to show w is a homotopy equivalence. Let

$$p : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \to Y_{n-1} \oplus Z_n,\ (x_1, y, x_2, z) \mapsto (fx_2 + y, z)$$

In tradition we need to prove it's a chain map.

$$\begin{aligned}
p\partial(x_1, y, x_2, z) &= p(\partial x_1, -fx_1 - \partial y, x_1 - \partial x_2, gfx_2 + gy + \partial z) \\
&= (-fx_1 - \partial y + fx_1 - f\partial x_2, gfx_2 + gy + \partial z) \\
&= (-\partial y - f\partial x_2, gfx_2 + gy + \partial z)
\end{aligned}$$

and $\partial p(x_1, y, x_2, z) = \partial(fx_2 + y, z) = (-\partial y - f\partial x_2, gfx_2 + gy + \partial z)$. Obviously $p \circ w = \mathrm{id}$ and we next prove $w \circ p \sim \mathrm{id}$.

$$(\mathrm{id} - wp)(x_1, y, x_2, z) = (x_!, -fx_2, x_2, 0)$$

Let

$$s_n : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \to X_{n-1} \oplus Y_n \oplus X_n \oplus Z_{n+1},\ (x_1, y, x_2, z) = (x_2, 0, 0, 0)$$

then

$$\begin{aligned}
&(\partial s + s\partial)(x_1, y, x_2, z)\\
=&\partial(x_2, 0, 0, 0) + s(\partial x_1, -fx_1 - \partial y, x_1 - \partial x_2, gfx_2 + gy + \partial z)\\
=&(x_1, -fx_2, x_2)
\end{aligned}$$

This proves w is a homotopy equivalence. □

In this long proof above, to find a new chain map or to find a chain homotopy is not difficult, since we can use the method of *undetermined coefficients*. And to check a given morphism is a homotopy equivalence is also not difficult. We often find its inverse, one composition equation in $\mathbf{Ch}(\mathcal{A})$ and the other in $\mathbf{K}(\mathcal{A})$. The strict equation will help us find the inverse and we only need to check the other part is up to homotopy.

Every distinguished triangle in $\mathbf{K}(\mathcal{A})$ corresponds to a short exact sequence in $\mathbf{Ch}(\mathcal{A})$ but conversely not every short exact sequence defines a distinguished triangle in $\mathbf{K}(\mathcal{A})$. However, the converse is true in derived categories and it's a part of the reason why derived categories can be used fully in homological algebra.

Example 3.43. Assume $\mathcal{A} = \mathbf{Ab}$. Given an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

its corresponding triangle

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[1]$$

is not distinguished, where we use an object in $\mathbf{Ab}$ to represent a chain complex centered at $n = 0$. Otherwise $w = 0$ and $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in $\mathbf{K}(\mathbf{Ab})$. But $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are all center at $n = 0$. Hence

$$\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\mathbb{Z}$$

in $\mathbf{Ab}$ which is impossible.

And note that Proposition 3.40 here says in a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

f is a quasi-isomorphism iff Z is exact. This is clear since a homotopy equivalence preserves homology groups and every distinguished triangle is homotopy equivalent to a standard one. Now we start to prove the following important theorem for derived categories.

Theorem 3.44. *Suppose $\mathcal{A}$ is abelian. Then $(\mathbf{K}(\mathcal{A}), \mathcal{W})$ is right/left Ore and right/left reversible.*

Proof. We only prove it's right Ore and right reversible since the other part is similar. Given a problem

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $q \in \mathcal{W}$. At first we obtain a distinguished triangle

$$Z \xrightarrow{q} Y \xrightarrow{u} U \longrightarrow Z[1]$$

from (TR2). Apply (TR2) again to the morphism $uf : X \to U$.

$$X \xrightarrow{uf} U \longrightarrow W[1] \xrightarrow{-t} X[1]$$

is distinguished. Then we have a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{t} & X & \xrightarrow{uf} & U & \longrightarrow & W[1] \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle g}\downarrow \\ Z & \xrightarrow[q]{} & Y & \xrightarrow[u]{} & U & \longrightarrow & Z[1] \end{array}$$

(TR4) tells us there will exist g making the diagram commutative. Since $q \in \mathcal{W}$, U is exact which means $t \in \mathcal{W}$. This is clear by Proposition 3.40.

Given a diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Y'$$

such that $qf = qg$ and $q \in \mathcal{W}$. Let $h = f - g \Rightarrow qh = 0$. By (TR2) we can find a distinguished triangle of the form

$$U \xrightarrow{u} Y \xrightarrow{q} Y' \longrightarrow U[1]$$

Since $q \in \mathcal{W}$, $U[1]$ is exact. Obviously U is hence exact. And by (TR4) we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ {\scriptstyle v}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle v}\downarrow \\ U & \xrightarrow[u]{} & Y & \xrightarrow[q]{} & Y' & \longrightarrow & U[1] \end{array}$$

satisfying $uv = h$. Apply (TR2) to $v : X \to U$ and we obtain a distinguished triangle

$$V \xrightarrow{s} X \xrightarrow{v} U \longrightarrow V[1]$$

Since U is exact, $s \in \mathcal{W}$. And we see $v \circ s = 0 \Rightarrow uvs = hs = 0 \Rightarrow fs = gs$. □

Now we define $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})[\mathcal{W}^{-1}]$. Let $\tau : \mathbf{Ch}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ be the localization functor. Then we can prove g is a quasi-isomorphism in $\mathbf{Ch}(\mathcal{A})$ iff $\tau(g)$ is an isomorphism in $\mathbf{D}(\mathcal{A})$.

Proof. Actually the part of "⇒" is clear and we prove "⇐". From Theorem 3.12, there will exist chain maps f, h such that gf and hg are quasi-isomorphisms. Then $g_{n*}f_{n*}$ and $h_{n*}g_{n*}$ are isomorphisms in $\mathbf{Ab}$. This means g_{n*} has a left inverse and a right inverse which implies it's an isomorphism. □

Although from Corollary 3.15 and Proposition 3.22 we conclude $\mathbf{D}(\mathcal{A})$ is additive, in general it's not abelian. In the following we will talk about an example but before this we should introduce some computations in homological algebra using derived categories.

Remark 3.45. In this remark we consider $\mathbf{D}(R)$ and R-modules are thought to be left R-modules. We will use the technique from model categories talked about in the previous chapter freely.

For any R-module M we compute its derived functor $\operatorname{Ext}^n_R(M, -)$. At first we look at the case $M = R$. Given a chain complex $X \in \mathbf{Ch}(R)$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \swarrow^{s} & \downarrow x & & & & \\ \cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots \end{array}$$

where the bottom complex is $X[-n]$. Then a chain map $R(0) \to X[-n]$ just means an element in $\ker \partial_n$. If we consider chain homotopies, $x \sim y$ if there exists some $s : R \to X_{n+1}$ which here is identified with $s(1)$, such that $x - y = \partial_{n+1}s$. This interpretation shows that

$$H_n(X) \cong \operatorname{Hom}_{\mathbf{K}(R)}(R, X[-n])$$

Actually we see $R(0)$ is bounded below and above and therefore $R(0)$ is cofibrant in $\mathbf{Ch}(R)$. And since every complex is fibrant, we conclude

$$\operatorname{Hom}_{\operatorname{Ho}(\mathbf{Ch}(R))}(R, X[-n]) \cong \operatorname{Hom}_{\mathbf{Ch}(R)}(R, X[-n])/\sim \cong \operatorname{Hom}_{\mathbf{K}(R)}(R, X[-n])$$

where $\operatorname{Ho}(\mathbf{Ch}(R))$ is just $\mathbf{D}(R)$. The last isomorphism comes from Example 2.34 which asserts right homotopies in the model category $\mathbf{Ch}(R)$ are just chain homotopies. Therefore

$$H_n(X) \cong \operatorname{Hom}_{\mathbf{D}(R)}(R, X[-n])$$

From this we see derived categories contain the information of homology naturally.

Now let us replace R by any module M.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \swarrow^{s} & \downarrow f & & & & \\ \cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots \end{array}$$

A chain map $f : M(0) \to X[-n]$ means a morphism $f : M \to X_n$ such that $\partial_n \circ f = 0$ which is just an element in $\ker \partial_{n*}$ where

$$\partial_{n*} : \mathrm{Hom}_R(M, X_n) \to \mathrm{Hom}_R(M, X_{n-1})$$

Any two morphisms $f,\ g$ are equivalent if there is some $s : M \to X_{n+1}$ such that $f - g = \partial_{n+1}s = \partial_{n+1*}(s)$. This implies

$$H_n(\mathrm{Hom}_R(M, X_\bullet)) \cong \mathrm{Hom}_{\mathbf{K}(R)}(M, X[-n]) \cong \mathrm{Hom}_{\mathbf{D}(R)}(M, X[-n])$$

If N is any other R-module, let

$$0 \longrightarrow N \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots$$

be its injective resolution and then

$$\mathrm{Ext}^n_R(M, N) \cong H_{-n}(\mathrm{Hom}_R(M, I_\bullet)) \cong \mathrm{Hom}_{\mathbf{D}(R)}(M, I_\bullet[n])$$

Moreover look at this diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I_0 & \longrightarrow & I_{-1} & \longrightarrow & \cdots \end{array}$$

The chain map induces isomorphic homology groups hence belonging to $\mathcal{W}$, which means in $\mathbf{D}(R)$, $N \cong I_\bullet$. And finally we obtain

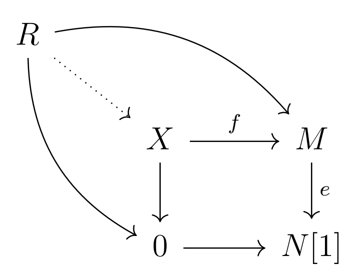
$$\mathrm{Ext}^n_R(M, N) \cong \mathrm{Hom}_{\mathbf{D}(R)}(M, N[n])$$

Example 3.46. In this example, we explain why $\mathbf{D}(R)$ and therefore $\mathbf{K}(R)$ are in general not abelian. Given any two modules $M,\ N$ such that $\mathrm{Ext}^1_R(M, N) \neq 0$. For instance, $R = \mathbb{Z}$ and $M = N = \mathbb{Z}/2\mathbb{Z}$, then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \neq 0$. From Remark 3.45

$$\mathrm{Ext}^1_R(M, N) \cong \mathrm{Hom}_{\mathbf{D}(R)}(M, N[1])$$

there will be a non-trivial element $e \in \mathrm{Hom}_{\mathbf{D}(R)}(M, N[1])$.

In the following we will prove e does not have a kernel. Actually, the kernel is just the pullback which is compatible with the translation functor. The compatibility actually needs some words. In a derived category the translation functor is also defined to be the shift of complexes. See Theorem 3.48 and Theorem 3.54.



We suppose such pullback exists in $\mathbf{D}(R)$. By the universal property of pullbacks we have the following exact sequence in $\mathbf{Ab}$

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(R, X[-n]) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(R, M[-n]) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(R, N[-n+1])$$

from Remark 3.45 which is just

$$0 \longrightarrow H_n(X) \longrightarrow H_n(M) \longrightarrow H_{n-1}(N)$$

This sequence is exact. But $H_{n-1}(N)$ is trivial except $H_0(N) = N$. But when $n = 1$, $H_1(M) = 0 \Rightarrow H_1(X) = 0$. Therefore f induces isomorphic homology groups. Since $\mathbf{K}(R)$ admits calculus of fractions according to Theorem 3.44, we may assume f is represented by $X \overset{s}{\leftarrow} Y \overset{g}{\rightarrow} M$ where $s \in \mathcal{W}$. Then the map $H_n(X) \to H_n(M)$ is just $g_* \circ s_*^{-1} : H_n(X) \to H_n(Y) \to H_n(M)$. This is an isomorphism. We conclude g is a quasi-isomorphism which implies f is an isomorphism in $\mathbf{D}(R)$. Then in the pullback diagram above $e = 0$ but we have supposed $0 \neq e \in \mathrm{Ext}^1_R(M, N)$. This means e does not have the kernel.

There is a triangulated category structure on $\mathbf{D}(\mathcal{A})$ as well which actually comes from $\mathbf{K}(\mathcal{A})$.

Definition 3.47. *A triangle is distinguished in $\mathbf{D}(\mathcal{A})$ is it's isomorphic to some standard one*

$$X \overset{f}{\rightarrow} Y \overset{\alpha_f}{\longrightarrow} M(f) \overset{\beta_f}{\longrightarrow} X[1]$$

in $\mathbf{D}(\mathcal{A})$.

Theorem 3.48. *If $\mathcal{A}$ is a triangulated category, then $\mathbf{D}(\mathcal{A})$ is an abelian category.*

Proof. This is a corollary of a more general Theorem 3.53. We only need to prove the functor $T = [-]$ is compatible with the triangulation (see Definition 3.52), but it's so clear by the definition of $T = [-]$. □

Sometimes we deal with subcategories of $\mathbf{Ch}(\mathcal{A})$ such as bounded, bounded below or bounded above chain complex category. For instance, in homological algebra we often talk about projective resolutions or injective resolutions which are not unbounded. Therefore here it's necessary to consider the localization for subcategories and we will use techniques from Section 3.3. We denote the category of bounded (resp. bounded below, resp. bounded above) chain complexes by $\mathbf{Ch}_b(\mathcal{A})$ (resp. $\mathbf{Ch}_+(\mathcal{A})$, resp. $\mathbf{Ch}_-(\mathcal{A})$). And its homotopy category is denoted by $\mathbf{K}_b(\mathcal{A})$ (resp. $\mathbf{K}_+(\mathcal{A})$, resp. $\mathbf{K}_-(\mathcal{A})$). The derived category is $\mathbf{D}_b(\mathcal{A})$ (resp. $\mathbf{D}_+(\mathcal{A})$, resp. $\mathbf{D}_-(\mathcal{A})$).

Theorem 3.49. *$\mathbf{D}_b(\mathcal{A})$ (resp. $\mathbf{D}_+(\mathcal{A})$, resp. $\mathbf{D}_-(\mathcal{A})$) is equivalent to the full subcategory of $\mathbf{D}(\mathcal{A})$ consisting of bounded (resp. bounded below, resp. bounded above) complexes.*

Proof. We will use Theorem 3.19 to prove this theorem. More generally we consider the category $\mathbf{Ch}_{\geq n}(\mathcal{A})$ consisting of objects Y of the form

$$\cdots \longrightarrow Y_{n+1} \longrightarrow Y_n \longrightarrow 0 \longrightarrow \cdots$$

Given a quasi-isomorphism $s : X \to Y$ with $Y \in \mathbf{Ch}_{\geq n}(\mathcal{A})$, we define

$$\tau_{\geq n}(X) := \cdots \longrightarrow X_{n+1} \longrightarrow \ker \partial_n \longrightarrow 0 \longrightarrow \cdots$$

belonging to $\mathbf{Ch}_{\geq n}(\mathcal{A})$. Since s is a quasi-isomorphism, $H_k(X) = 0$ for $k \leq n-1$. Then the inclusion $\tau_{\geq n}X \hookrightarrow X$ is a quasi-isomorphism and therefore $\tau_{\geq n}X \hookrightarrow X \xrightarrow{s} Y$ is a quasi-isomorphism in $\mathbf{Ch}_{\geq n}(\mathcal{A})$. Then Theorem 3.19 implies $\mathbf{K}_{\geq n}(\mathcal{A})$ is a localizing subcategory of $\mathbf{K}(\mathcal{A})$.

Dually suppose $s : X \to Y$ is a quasi-isomorphism with $X \in \mathbf{Ch}_{\leq n}(\mathcal{A})$. Then $H_k(Y) = 0$ for $k \geq n+1$. Define

$$\tau_{\leq n}(Y) := \cdots \longrightarrow 0 \longrightarrow \operatorname{coker} \partial_{n+1} = Y_n/\operatorname{im} \partial_{n+1} \to Y_{n-1} \longrightarrow \cdots$$

Then there is a natural quasi-isomorphism $Y \to \tau_{\leq n}Y$ and $X \to Y \to \tau_{\leq n}Y$ is a quasi-isomorphism belonging to $\mathbf{Ch}_{\leq n}(\mathcal{A})$. Then $\mathbf{K}_{\leq n}(\mathcal{A})$ is also a localizing subcategory of $\mathbf{K}(\mathcal{A})$.

For bounded, bounded below or bounded above complexes we can deal with them similarly. □

To finish the section we complete the work above Example 3.43 and prove every short exact sequence in $\mathcal{A}$ will give a distinguished triangle in $\mathbf{D}(\mathcal{A})$. To do this let us introduce the concept of *mapping cylinder* in $\mathbf{Ch}(\mathcal{A})$.

Definition 3.50. *For a chain map $f : X \to Y$, the* ***mapping cylinder*** *is defined to be*

$$\operatorname{Cyl}(f)_n := X_n \oplus X_{n-1} \oplus Y_n,\ \partial : (x_1, x_2, y) \mapsto (\partial x_1 - x_2, -\partial x_2, f x_2 + \partial y)$$

Checking the three positions pointwise, it's clear that the differential operator ∂ is well defined.

Theorem 3.51. *Given an abelian category $\mathcal{A}$, every short exact sequence in $\mathbf{Ch}(\mathcal{A})$*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

gives a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

in $\mathbf{D}(\mathcal{A})$.

Proof. Firstly we study the mapping cylinder further. We have an inclusion $\iota : X \hookrightarrow \operatorname{Cyl}(f),\ x \mapsto (x, 0, 0)$ and it's obvious to see it's a chain map. And we have a projection

$$\pi : \operatorname{Cyl}(f) \to M(f),\ (x_1, x_2, y) \mapsto (x_2, y)$$

To prove it's a chain map, we compute

$$\partial\pi(x_1, x_2, y) = \partial(x_2, y) = (-\partial x_2, f x_2, \partial y) = \pi\partial(x_1, x_2, y)$$

Then we obtain an exact sequence

$$0 \longrightarrow X \xleftarrow{\iota} \operatorname{Cyl}(f) \xrightarrow{\pi} M(f) \longrightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & Y & \xrightarrow{\alpha_f} & M(f) & \xrightarrow{\beta_f} & X[1] & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\mathrm{id}} & & & & \\
0 & \longrightarrow & X & \xrightarrow{\iota} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & M(f) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\tau} & & & & & & \\
 & & X & \xrightarrow[f]{} & Y & & & & & &
\end{array}$$

where

$$\sigma_n : Y_n \to X_n \oplus X_{n-1} \oplus Y_n,\ y \mapsto (0, 0, y)$$

and

$$\tau_n : X_n \oplus X_{n-1} \oplus Y_n \to Y_n,\ (x_1, x_2, y) \mapsto fx_1 + y$$

It's clear that σ is a chain map but for τ it need some words.

$$\begin{aligned}
\tau\partial(x_1, x_2, y) &= \tau(\partial x_1 - x_2, -\partial x_2, fx_2 + \partial y) \\
&= f\partial x_1 - fx_2 + fx_2 + \partial y \\
&= f\partial x_1 + \partial y \\
&= \partial fx_1 + \partial y \\
&= \partial\tau(x_1, x_2, y)
\end{aligned}$$

Next we will prove τ and σ are homotopy equivalences. Firstly it's clear $\tau \circ \sigma = \mathrm{id}_Y$. Therefore it's enough to prove $\sigma \circ \tau \sim \mathrm{id}_{\mathrm{Cyl}(f)}$.

$$\sigma \circ \tau : X_n \oplus X_{n-1} \oplus Y_n \to X_n \oplus X_{n-1} \oplus Y_n,\ (x_1, x_2, y) \mapsto (0, 0, fx_1 + y)$$

then $(\sigma\tau - \mathrm{id})(x_1, x_2, y) = (-x_1, -x_2, fx_1)$. Define

$$s_n : X_n \oplus X_{n-1} \oplus Y_n \to X_{n+1} \oplus X_n \oplus Y_{n+1},\ (x_1, x_2, y) \mapsto (0, x, 0)$$

We have

$$\begin{aligned}
(\partial s + s\partial)(x_1, x_2, y) &= (-x_1, -\partial x_!, fx_1) + (0, \partial x_1 - x_2, 0) \\
&= (-x_1, -x_2, f_1)
\end{aligned}$$

Hence $\sigma \circ \tau = \mathrm{id}$ in $\mathbf{K}(\mathcal{A})$. Especially they are quasi-isomorphisms. If we give a short exact sequence in $\mathbf{Ch}(\mathcal{A})$

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we have the commutative diagram by diagram chasing

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & M(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\tau} & & \vdots{\scriptstyle\gamma\ \exists!} & & \\
0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0
\end{array}$$

Since τ and id are quasi-isomorphisms, applying the 5-lemma (at the leveal of $\mathbf{Ab}$) to the induced long exact homology sequences we conclude γ is a quasi-isomorphism hence an isomorphism in $\mathbf{D}(\mathcal{A})$. Moreover in $\mathbf{D}(\mathcal{A})$ we have the following isomorphic triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha_f} & M(f) & \xrightarrow{\beta_f} & X[1] \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{\iota} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & M(f) & \xrightarrow{\beta_f} & X[1] \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \tau} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[\beta_f\gamma^{-1}]{} & X[1] \end{array}$$

which proves

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

is distinguished. □

Note that Theorem 3.51 is not ture in $\mathbf{K}(\mathcal{A})$ (see Example 3.43) because such γ may not be a homotopy equivalence.

3.6 Verdier Quotient in General

In this section we will talk about the abstract theory of the localization for triangulated categories as a generalization of the case $\mathbf{K}(\mathcal{A})$.

Recall for a category $\mathcal{C}$, a multiplicative system $\mathcal{W}$ is right Ore and right reversible. See Definition 3.8.

Definition 3.52. *For a triangulated category $(\mathcal{A}, T)$, a multiplicative system $\mathcal{W}$ is compatible with the triangulation if given $f \in \mathcal{W}$, $T^n f \in \mathcal{W}$ of all $n \in \mathbb{Z}$.*

Theorem 3.53. *Let $(\mathcal{A}, T)$ be a triangulated category with $\mathcal{W}$ a multiplicative system which is compatible with the triangulation. Then $\mathcal{A}\mathcal{W}^{-1}$ carries the unique triangulated structure such that the localization functor $\tau : \mathcal{A} \to \mathcal{A}\mathcal{W}^{-1}$ is exact.*

We should explain the exactness between two triangulated categories first.

Definition 3.54. *An exact functor $F : (\mathcal{A}, T) \to (\mathcal{A}', T')$ of trnaigulated categories is additive together with natural isomorphisms*

$$\varphi_X : F(TX) \xrightarrow{\sim} T'F(X)$$

such that for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

in $\mathcal{A}$,

$$F(X) \xrightarrow{F(u)} Y \xrightarrow{F(v)} F(Z) \xrightarrow{\varphi_X \circ F(w)} T'F(X)$$

is distinguished in $\mathcal{A}'$.

Proof of Theorem 3.53. We show what's the translation functor in $\mathcal{A}\mathcal{W}^{-1}$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tau} & \mathcal{A}\mathcal{W}^{-1} \\ {\scriptstyle T}\downarrow & & \\ \mathcal{A} & & {\scriptstyle T' \ \exists!} \swarrow \\ {\scriptstyle \tau}\downarrow & & \\ \mathcal{A}\mathcal{W}^{-1} & & \end{array}$$

T' exists since T respects morphisms in $\mathcal{W}$. Replace T in this diagram by T^{-1} we obtain $T_2' : \mathcal{A}\mathcal{W}^{-1} \to \mathcal{A}\mathcal{W}^{-1}$. They satisfies

$$\begin{cases} T' \circ \tau = \tau \circ T \\ T_2' \circ \tau = \tau \circ T^{-1} \end{cases}$$

Then $T'T_2' \circ \tau = T' \circ \tau T^{-1} \circ \tau \circ T \circ T^{-1} = \tau$. By the universal property of localization, we see $T'T_2' = \mathrm{id}_{\mathcal{A}\mathcal{W}^{-1}}$. It's clear $T_2' \circ T' = \mathrm{id}$ as well. T' is just the translation functor for $\mathcal{A}\mathcal{W}^{-1}$. The natural isomorphisms φ_X in Definition 3.54 are actually identities.

A triangle is distinguished in $\mathcal{A}\mathcal{W}^{-1}$ if it's isomorphic to a standard one coming from $\mathcal{A}$ via the localization functor $\tau : \mathcal{A} \to \mathcal{A}\mathcal{W}^{-1}$.

(TR0) and (TR1) are clear.

(TR2). Given a morphism $X \xleftarrow{s} A \xrightarrow{f} Y$ in $\mathrm{Hom}_{\mathcal{A}\mathcal{W}^{-1}}(X, Y)$ where $s \in \mathcal{W}$. Then

$$\begin{array}{ccccccc} A & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & TA \\ {\scriptstyle s}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ X & \xrightarrow[fs^{-1}]{} & Y & \longrightarrow & Z & \longrightarrow & TX \end{array}$$

Note that s is an isomorphism in $\mathcal{A}\mathcal{W}^{-1}$.

(TR3), (TR4) and (TR5) are all easy to see since we can just check these axioms at the level of standard triangles. And then the triangulated structure comes from $\mathcal{A}$.

Finally we need to prove this triangulated structure is unique to make τ exact. Suppose τ is exact and given any distinguished triangle

$$X \xrightarrow{fs^{-1}} Y \longrightarrow Z \longrightarrow TX$$

we prove it's isomorphic to a standard one. In fact

$$\begin{array}{ccccccc} A & \xrightarrow{f} & Y & \longrightarrow & C & \longrightarrow & TA \\ {\scriptstyle s}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \exists}\vdots & & \downarrow \\ X & \xrightarrow[fs^{-1}]{} & Y & \longrightarrow & Z & \longrightarrow & TX \end{array}$$

where the above triangle comes from $\mathcal{A}$ and the dotted arrow comes from (TR4) which is an isomorphism by Theorem 3.30. □

Definition 3.55. *Let $(\mathcal{A}, T)$ be a triangulated category. A full additive subcategory $\mathcal{C}$ is called triangulated subcategory if every object in $\mathcal{A}$ isomorphic to an object in $\mathcal{C}$ is in $\mathcal{C}$ and $\mathcal{C}$ satisfies*

(TS1) For any $X \in \mathrm{Ob}(\mathcal{C})$, $T^n X \in \mathrm{Ob}(\mathcal{C})$ for all $n \in \mathbb{Z}$.

(TS2) If

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX$$

is distinguished in $\mathcal{A}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$, then $Z \in \mathcal{C}$.

(TS2) is equivalent to say for a distinguished triangle if two of $X,\ Y,\ Z$ are in $\mathcal{C}$, then so is the remainder. These two axioms make $\mathcal{C}$ be a triangulated category and the triangulated structure comes from $\mathcal{A}$. For a triangulated subcategory $\mathcal{C}$, we define $\mathcal{W}_\mathcal{C} \subseteq \mathrm{Mor}(\mathcal{A})$ such that $f \in \mathcal{W}_\mathcal{C}$ iff there is some distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

with $Z \in \mathrm{Ob}(\mathcal{C})$. From (TR4) and triangulated 5-lemma, Z will be unique up to isomorphism.

Lemma 3.56. *If $f \in \mathcal{W}_\mathcal{C}$, then $Tf \in \mathcal{W}_\mathcal{C}$.*

Proof. Suppose $f \in \mathcal{W}_\mathcal{C}$ and we have isomorphic diagrams

$$\begin{array}{ccccccc} X & \xrightarrow{-f} & Y & \xrightarrow{g} & Z & \xrightarrow{-h} & TX \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & TX \end{array}$$

which implies $-f \in \mathcal{W}_\mathcal{C}$. By (TR3)

$$TX \xrightarrow{-Tf} TY \xrightarrow{-Tg} TZ \xrightarrow{-Th} T^2X$$

$Z \in \mathrm{Ob}(\mathcal{C}) \Rightarrow TZ \in \mathrm{Ob}(\mathcal{C})$. Then $-Tf \in \mathcal{W}_\mathcal{C}$ and $Tf \in \mathcal{W}_\mathcal{C}$ according to the statement above. □

Lemma 3.57. *Every isomorphism $f : X \to Y$ is in $\mathcal{W}_\mathcal{C}$*

Proof. Assume $g : Y \to X$ is the inverse of f.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow \\ X & \xrightarrow[\mathrm{id}]{} & X & \longrightarrow & 0 & \longrightarrow & TX \end{array}$$

The above triangle is obtained by (TR2). From Theorem 3.30, $Z \xrightarrow{\cong} 0$. Then

$$X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow TX$$

is distinguished. Since $\mathcal{C}$ is an additive subcategory, $0 \in \mathrm{Ob}(\mathcal{C})$. Then $f \in \mathcal{W}_\mathcal{C}$. □

Lemma 3.58. *$\mathcal{W}_{\mathcal{C}}$ satisfies the property of two out of three, which means given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ if any two of $f,\ g$ and gf are in $\mathcal{W}_{\mathcal{C}}$, then so is the remainder.*

Proof. (TR5) tells us there is the following diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & TX \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \longrightarrow & TX \\
{\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow[g]{} & Z & \longrightarrow & X' & \longrightarrow & TY
\end{array}$$

With any two of $f,\ g$ and gf belonging to $\mathcal{W}_{\mathcal{C}}$, we can choose corresponding $X',\ Y'$ or Z' in $\mathcal{C}$, and then (TR5) implies there is a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow TZ'$$

with two objects in $\mathcal{C}$. Since $\mathcal{C}$ is a triangulated subcategory, (TS2) implies the remainder is also in $\mathcal{C}$. Then the third morphism will be in $\mathcal{W}_{\mathcal{C}}$ by definition. □

Theorem 3.59. *$\mathcal{W}_{\mathcal{C}}$ is right/left Ore and right/left reversible.*

Proof. Given a problem

$$\begin{array}{ccc}
 & & Z \\
 & & \downarrow{\scriptstyle q} \\
X & \xrightarrow[f]{} & Y
\end{array}$$

with $q \in \mathcal{W}_{\mathcal{C}}$, there will exist a distinguished triangle

$$Z \xrightarrow{q} Y \xrightarrow{u} A \longrightarrow TZ$$

with $A \in \mathcal{C}$. Applying (TR2) to $uf : X \to A$ we obtain a distinguished triangle

$$X \xrightarrow{uf} A \longrightarrow TB \xrightarrow{-Tt} TX$$

and from (TR3) this triangle becomes a new distinguished one

$$B \xrightarrow{t} X \xrightarrow{uf} A \longrightarrow TB$$

Since $A \in \mathcal{C}$, $t \in \mathcal{W}_{\mathcal{C}}$. By (TR4) there will exist the dotted arrow g making the diagram commuative.

$$\begin{array}{ccccccc}
B & \xrightarrow{t} & X & \xrightarrow{uf} & A & \longrightarrow & TB \\
{\scriptstyle g}\vdots & & {\scriptstyle f}\downarrow & & \| & & \vdots{\scriptstyle Tg} \\
Z & \xrightarrow[q]{} & Y & \xrightarrow[u]{} & A & \longrightarrow & TZ
\end{array}$$

For the second part of the proof, we give the diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Z$$

such that $qf = qg$ and $q \in \mathcal{W}_\mathcal{C}$. Let $h = f - g$ then $qh = 0$. Since $q \in \mathcal{W}_\mathcal{C}$, we have a dinstinguished triangle

$$Y \xrightarrow{q} Z \longrightarrow A \longrightarrow TY$$

with $A \in \mathcal{C}$, by (TR2) this induces a new one

$$T^{-1}A \xrightarrow{u} Y \xrightarrow{q} Z \longrightarrow Z$$

Finally from (TR4),

$$\begin{array}{ccccccc} X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & TX \\ {\scriptstyle v}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow & & \downarrow \\ T^{-1}A & \underset{u}{\longrightarrow} & Y & \underset{q}{\longrightarrow} & Z & \longrightarrow & A \end{array}$$

Embed v in the distinguished triangle

$$V \xrightarrow{s} X \xrightarrow{v} T^{-1}A \longrightarrow TV$$

where $T^{-1}A \in \mathcal{C}$. Then $vs = 0 \Rightarrow 0 = uvs = hs \Rightarrow fs = gs$. □

Then $\mathcal{W}_\mathcal{C}$ will be a multiplicative compatible with the triangulation.

Application 3.60. In the case of derived categories $\mathbf{D}(\mathcal{A})$, the triangulated subcategory $\mathcal{C}$ consists of all exact complexes and from Proposition 3.40 $\mathcal{W}_\mathcal{C}$ consists of all quasi-isomorphisms.

We denote the localization category $\mathcal{A}[\mathcal{W}_\mathcal{C}^{-1}]$ by $\mathcal{A}/\mathcal{C}$ and in the following readers will why we use this symbol (see Theorem 3.65).

Definition 3.61. *If $F : \mathcal{A} \to \mathcal{A}'$ is an exact functor between triangulated categories (see Definition 3.54), the **kernel** of F is defined to be the full subcategory $\ker F$ of $\mathcal{A}$ such that*

$$X \in \mathrm{Ob}(\ker F) \text{ iff } F(X) \cong 0 \text{ in } \mathcal{A}'$$

Lemma 3.62. *$\ker F$ defined above is a triangulated subcateory in the sense of Definition 3.55.*

Proof. Since an additive functor between additive categories preserve direct sums, $\ker F$ is a full additive subcategory and it's closed under isomorphism.

(TS1). Let $X \in \ker F \Rightarrow FX \cong 0$. And via natural isomorphisms φ, $F(TX) \xrightarrow{\sim} T(FX) \cong 0$. Then $TX \in \ker F$. And note that $0 \cong F \circ T(T^{-1}X) \xrightarrow{\cong} T(FT^{-1}X) \Rightarrow T(FT^{-1}X) \cong 0$. Hence $FT^{-1}X \cong 0$ and $T^{-1}X \cong \ker F$.

(TS2). Given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$

with $X,\ Y \in \ker F$. Apply F to this diagram and we will obtain a distinguished triangle in $\mathcal{A}'$ with $FX,\ FY,\ TFX \cong 0$. From (TR4) and triangulated 5-lemma, there will exist an isomorphism $F(Z) \to 0$ making this diagram isomorphic to the zero triangle. Hence $Z \in \ker F$. □

Definition 3.63. *A triangulated subcategory $\mathcal{C}$ of $\mathcal{A}$ is called **thick** if it contains all direct summand of its objects, which means if $X \oplus Y \in \mathrm{Ob}(\mathcal{C})$ then $X,\ Y \in \mathrm{Ob}(\mathcal{C})$.*

Lemma 3.64. $\ker F$ *is thick.*

Proof. If $X \oplus Y \in \mathrm{Ob}(\ker F)$, $F(X) \oplus F(Y) \cong 0$ and then $F(X) \cong F(Y) \cong 0$. □

Theorem 3.65. *Let $\mathcal{A}$ be a triangulated category with $\mathcal{C}$ a triangulated subcategory of $\mathcal{A}$ not necessarily thick. Then there exists a universal exact functor $\tau : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ with $\mathcal{C} \subseteq \ker \tau$. Here τ is universal in the sense that given any exact functor $F : \mathcal{A} \to \mathcal{A}'$ between triangulated categories with $\mathcal{C} \subseteq \ker F$, then there will exist the unique funcotr $\theta : \mathcal{A}/\mathcal{C} \to \mathcal{A}'$ saitsfying $\theta \circ \tau = F$.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tau} & \mathcal{A}/\mathcal{C} \\ {\scriptstyle F}\downarrow & \swarrow & \\ \mathcal{A}' & & \end{array}$$

Proof. The universal functor is just the localization functor τ and we prove it satisfies the universal property for kernels. To prove this we only need to prove an exact functor F with $\mathcal{C} \subseteq \ker F$ sends morphisms in $\mathcal{W}_S$ to isomorphisms. But at first we should check $\mathcal{C} \subseteq \ker \tau$.

Given $X \in \mathrm{Ob}(\mathcal{C})$, by (TR1) and (TR3)

$$X \longrightarrow 0 \longrightarrow TX \xrightarrow{-\mathrm{id}} TX$$

is distinguished. Since $TX \in \mathrm{Ob}(\mathcal{C})$, $X \to 0$ belongs to $\mathcal{W}_\mathcal{C}$. Hence $\tau(X) \to 0$ is an isomorphism in $\mathcal{A}/\mathcal{C}$, which means $X \in \ker \tau$.

Given any exact functor $F : \mathcal{A} \to \mathcal{A}'$ with $\mathcal{C} \subseteq \ker F$, suppose $q : X \to Y$ belongs to $\mathcal{W}_\mathcal{C}$ and then there will be a distinguished triangle

$$X \xrightarrow{q} Y \longrightarrow Z \longrightarrow TX$$

with $Z \in \mathcal{C}$.

$$\begin{array}{ccccccc} FX & \xrightarrow{Fq} & FY & \longrightarrow & FZ & \longrightarrow & TFX \\ \| & & \| & & \cong\downarrow & & \| \\ FX & \xrightarrow{Fq} & FY & \longrightarrow & 0 & \longrightarrow & TFX \\ \| & & \cong\downarrow & & \| & & \| \\ FX & \xrightarrow{\mathrm{id}} & FX & \longrightarrow & 0 & \longrightarrow & TFX \end{array}$$

From isomorphisms between the top triangle and the middle triangle, we see the middle one is distinguished. Then by (TR4) there will exist an isomorphism $FY \xrightarrow{\sim} FX$ making the diagram commutative. This proves Fq is an isomorphism. Then since τ is the localization functor, the unique functor θ exists. □

In this theorem, $\mathcal{A}/\mathcal{C} := \mathcal{A}[\mathcal{W}_{\mathcal{C}}^{-1}]$ is called the *Verdier quotient* and τ is the Verdier localization map.

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