

Perverse Sheaves I Talk 10 (First Half)

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Abstract

These notes about perverse t -structures are written for the talk 10 of the MSc. Seminar on perverse sheaves held by Dr. Stefan Dawydiak in the University of Bonn and the seminar page is in <https://www.math.uni-bonn.de/people/dawydiak/perverse.html>.

Contents

1 Perverse t -Structures

1

1 Perverse t -Structures

In these notes, we always suppose our base field $k = \mathbb{C}$ is the field of complex numbers.

Let X be a variety over k . Recall $D(X, k)$ or simply $D(X)$ is the *derived category of sheaves of k -vector spaces on X^{an}* where X^{an} means it's with the analytic topology. $D_c^b(X) \subseteq D(X)$ is the full subcategory consisting of bounded complexes \mathcal{F} whose cohomological sheaves $H^i(\mathcal{F})$ are *constructible* for all i . Next we define two full subcategories of $D_c^b(X)$.

Definition 1.1. The full subcategory ${}^p D_c^{\leq 0}(X)$ of $D_c^b(X)$ consists of bounded constructible complex $\mathcal{F} \in D_c^b(X)$ such that

$$\dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$$

for all i .

And on the other hand we want to define ${}^p D_c^{\geq 0}(X)$, so that we will obtain

$$\operatorname{Perv}(X) := {}^p D_c^{\leq 0}(X) \cap {}^p D_c^{\geq 0}(X)$$

the category of *perverse sheaves*.

Verdier Duality For a variety $p_X : X \rightarrow \operatorname{pt}$ over k , the *dualizing complex* is $\omega_X = p_X^! k_{\operatorname{pt}}$ and from [1, Cor. 2.2.10] we see if X is smooth of dimension n , then $\omega_X \simeq k_X[2n]$ is centered at degree $-2n$.

The *Verdier duality* functor is defined to be

$$\mathbb{D} : D_c^b(X)^{op} \rightarrow D_c^b(X), \mathcal{F} \mapsto \mathbb{R}\mathcal{H}om(\mathcal{F}, \omega_X)$$

and the evaluation map $\mathcal{F} \rightarrow \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an equivalence.

Definition 1.2. The full subcategory ${}^p D_c^{\geq 0}(X)$ of $D_c^b(X)$ consists of bounded constructible complex $\mathcal{F} \in D_c^b(X)$ such that

$$\dim \operatorname{supp} H^i(\mathbb{D}\mathcal{F}) \leq -i$$

for all i . This means $\mathcal{F} \in {}^p D_c^{\geq 0}(X)$ if and only if $\mathbb{D}\mathcal{F} \in {}^p D_c^{\leq 0}(X)$.

Definition 1.3. The category of *perverse sheaves* is defined to be $\text{Perv}(X) := {}^pD_c^{\leq 0}(X) \cap {}^pD_c^{\geq 0}(X)$.

Remark 1.4. The equivalence $\mathbb{D}\mathcal{F} \simeq \mathcal{F}$ implies the Verdier duality functor exchanges ${}^pD_c^{\leq 0}(X)$ and ${}^pD_c^{\geq 0}(X)$, so it preserves $\text{Perv}(X)$.

Remark 1.5 (bounded t -structure). We will see later that the pair $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ defines a bounded t -structure in the triangulated category $D_c^b(X)$ ¹ in the sense of last talk and it's called the *perverse t -structure*.

Note that to see the perverse t -structure is *bounded* we need to show for any $\mathcal{F} \in D_c^b(X)$ there exists a positive integer $n > 0$ such that

$$\mathcal{F} \in {}^pD_c^{\leq n}(X) := {}^pD_c^{\leq 0}(X)[-n] \text{ and } \mathcal{F} \in {}^pD_c^{\geq -n}(X) := {}^pD_c^{\geq 0}(X)[n]$$

$\mathcal{F} \in {}^pD_c^{\leq n}(X)$ is equivalent to $\dim \text{supp} H^{i+n}(\mathcal{F}) \leq -i$. Since \mathcal{F} is a bounded complex, there is an integer $m > 0$ such that for all $i \geq m$, $H^i(\mathcal{F}) = 0$. Let $m' = \text{Max}(\dim \text{supp} H^i(\mathcal{F}))$. Then for all $n \geq m + m'$, we will have $\mathcal{F} \in {}^pD_c^{\leq n}(X)$. A similar argument can be applied to $\mathbb{D}\mathcal{F}$. So this proves the perverse t -structure is bounded.

Notation 1.6. From [1, Thm. A.7.8], for a t -structure in a triangulated category, its heart is an abelian category. Therefore $\text{Perv}(X)$ is actually an abelian category. And we have truncation functors

$$p_{\tau}^{\leq n} : D_c^b(X) \rightarrow {}^pD_c^{\leq n}(X) = {}^pD_c^{\leq 0}(X)[-n], \quad p_{\tau}^{\geq n} : D_c^b(X) \rightarrow {}^pD_c^{\geq n}(X) = {}^pD_c^{\geq 0}(X)[-n]$$

where $p_{\tau}^{\leq n}$ is right adjoint to ${}^pD_c^{\leq n}(X) \hookrightarrow D_c^b(X)$ and $p_{\tau}^{\geq n}$ is left adjoint to ${}^pD_c^{\geq n}(X) \hookrightarrow D_c^b(X)$.

Also for every $n \in \mathbb{Z}$, we have the cohomology functor

$${}^pH^n : D_c^b(X) \rightarrow \text{Perv}(X), \quad \mathcal{F} \mapsto p_{\tau}^{\leq 0} p_{\tau}^{\geq 0}(\mathcal{F}[n])$$

Good Stratifications

Definition 1.7 (good stratification). Let X be a variety with a stratification $(X_s)_{s \in \mathcal{S}}$ and for every $s \in \mathcal{S}$, $j_s : X_s \hookrightarrow X$ is the inclusion map. This stratification is called a *good stratification* if for every $s \in \mathcal{S}$ and every local system of finite type \mathcal{L} on X_s , we have $j_{s*} \mathcal{L} \in D_{\mathcal{S}}^b(X)$.

Remark 1.8. Every stratification can be refined to be a good one.

Example 1.9 (Normal Crossings Stratification). A Weil divisor $D = \sum_i D_i \subseteq X$ on a smooth variety X of dimension n is a *simple normal crossing* if every component D_i is smooth and for every point $p \in X$ a local equation of D is $x_1 \cdots x_r$ for independent local parameters $x_i \in \mathcal{O}_{X,p}$ with $r \leq n$.

Let $Z \subseteq X$ be a simple normal crossing in a smooth variety X of dimension n with irreducible components Z_1, \dots, Z_k . Then we can partition X into subvarieties labelled by subsets $J \subseteq \{1, \dots, k\}$ such that

$$X_J := \{x \in X \mid x \in Z_j \text{ if and only if } j \in J\} = \{x \in X \mid I(x) = J\}$$

where $I(x) = \{i \mid x \in Z_i\}$. For example if $J = \{1\}$, then $X_J = Z_1 \setminus (Z_2 \cup \dots \cup Z_k)$. For a general subset J , we have $X_J = (\cap_{j \in J} Z_j) \setminus (\cup_{i \notin J} Z_i)$. And by definition $X_{\emptyset} = X \setminus Z$. The closure of X_J will be $Z_J := \cap_{j \in J} Z_j$.

Non-empty subsets in $(X_J)_{J \subseteq \{1, \dots, k\}}$ gives a stratification of X , which is called the *normal crossings stratification*. From [1, Lem. 2.4.2], this stratification is good.

For a good stratification $(X_s)_{s \in \mathcal{S}}$, the induced t -structure on $D_{\mathcal{S}}^b(X)$ is given by

$$\begin{aligned} {}^pD_{\mathcal{S}}^{\leq 0}(X) &:= {}^pD_c^{\leq 0}(X) \cap D_{\mathcal{S}}^b(X) \\ {}^pD_{\mathcal{S}}^{\geq 0}(X) &:= {}^pD_c^{\geq 0}(X) \cap D_{\mathcal{S}}^b(X) \end{aligned}$$

The heart is denoted by $\text{Perv}_{\mathcal{S}}(X) = {}^pD_{\mathcal{S}}^{\leq 0}(X) \cap {}^pD_{\mathcal{S}}^{\geq 0}(X)$.

¹ [1, Thm. 3.1.9]

Remark 1.10. [1, Ex. 2.8.2] says if \mathcal{S} is a good stratification, the Verdier duality functor has a restriction $\mathbb{D} : D_{\mathcal{S}}^b(X)^{op} \rightarrow D_{\mathcal{S}}^b(X)$. So in this case \mathbb{D} preserves $\text{Perv}_{\mathcal{S}}(X)$.

Theorem 1.11. *Let X be a variety over k .*

- The perverse t -structure $({}^p D_c^{\leq 0}(X), {}^p D_c^{\geq 0}(X))$ is a t -structure in the triangulated category $D_c^b(X)$.
- Moreover if $(X_s)_{s \in \mathcal{S}}$ is a good stratification for X , then $({}^p D_{\mathcal{S}}^{\leq 0}(X), {}^p D_{\mathcal{S}}^{\geq 0}(X))$ gives a t -structure in $D_{\mathcal{S}}^b(X)$.

Sketch of the proof. We need to check three conditions for $({}^p D_c^{\leq 0}(X), {}^p D_c^{\geq 0}(X))$ to be a t -structure in $D_c^b(X)$.

- (1). ${}^p D_c^{\leq -1}(X) \subseteq {}^p D_c^{\leq 0}(X)$ and ${}^p D_c^{\geq 0}(X) \subseteq {}^p D_c^{\geq -1}(X)$.
- (2). If $\mathcal{F} \in {}^p D_c^{\leq -1}(X)$ and $\mathcal{G} \in {}^p D_c^{\geq 0}(X)$, then $\text{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}) = 0$.
- (3). For any $\mathcal{F} \in D_c^b(X)$, there is a distinguished triangle

$$\mathcal{H} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

with $\mathcal{H} \in {}^p D_c^{\leq -1}(X)$ and $\mathcal{G} \in {}^p D_c^{\geq 0}(X)$.

The first condition is just by definition, but note that $\mathbb{D}(\mathcal{F}[-1]) = (\mathbb{D}\mathcal{F})[1]$. Conditions of (2) and (3) will be checked in the second half part of this talk. □

Shifted Local Systems Now let's discuss relations between perverse sheaves and local systems.

If we choose the stratification $(X_s)_{s \in \mathcal{S}}$ to be the trivial stratification i.e. $X_s = X$, then $D_{\mathcal{S}}^b(X)$ will be $D_{\text{locf}}^b(X)$ the derived category consisting of complexes whose cohomologies are local systems of finite type.

Proposition 1.12. *Let X be a smooth connected variety of dimension n . Then we have*

$$\begin{aligned} {}^p D_{\text{locf}}^{\leq 0}(X) &= D_{\text{locf}}^{\leq -n}(X) \\ {}^p D_{\text{locf}}^{\geq 0}(X) &= D_{\text{locf}}^{\geq -n}(X) \end{aligned}$$

So that if we let \mathcal{S} be the trivial stratification on X , then

$$\text{Perv}_{\mathcal{S}}(X) = \text{Loc}^{\text{ft}}(X)[n]$$

In particular, if $X = \text{pt}$, $\text{Perv}_{\mathcal{S}}(\text{pt}) = \text{Vect}_k^{\text{ft}}$.

In this case, objects in $\text{Loc}^{\text{ft}}(X)[n]$ are called *shifted local systems*.

Proof. At first it's easy to see $D_{\text{locf}}^{\leq -n}(X) \subseteq {}^p D_{\text{locf}}^{\leq 0}(X)$ because $\text{supp} H^i(\mathcal{F})$ has at most dimension n . For the other side, it's clear for an object $\mathcal{F} \in D_{\text{locf}}^b(X)$, its cohomology sheaf $H^i(\mathcal{F})$, if it's non-zero, is a local system of finite type which means $\dim \text{supp} H^i(\mathcal{F}) = \dim X = n$ because X is connected. So if $\mathcal{F} \in {}^p D_{\text{locf}}^{\leq 0}(X)$, $\dim \text{supp} H^i(\mathcal{F}) \leq -i$ and then when $H^i(\mathcal{F}) \neq 0$, $n \leq -i \Leftrightarrow i \leq -n$. This proves ${}^p D_{\text{locf}}^{\leq 0}(X) = D_{\text{locf}}^{\leq -n}(X)$.

As for ${}^p D_{\text{locf}}^{\geq 0}(X) = D_{\text{locf}}^{\geq -n}(X)$, firstly notice that

$$\mathcal{F} \in D_{\text{locf}}^{\geq -n}(X) \text{ iff } \mathcal{F}_x \in D^{\geq -n}(\text{Vect}_k) \text{ for all } x \in X$$

Let $i_x : x \hookrightarrow X$ be the embedding. Then $\mathcal{F}_x = i_x^* \mathcal{F}$. The t -exactness of i_x^* implies $H^i(i_x^* \mathcal{F}) \simeq i_x^* H^i(\mathcal{F})$. So if $H^i(\mathcal{F}) = 0$, then $H^i(\mathcal{F}_x) = 0$. Conversely if $H^i(\mathcal{F}) \neq 0$, since $H^i(\mathcal{F})$ is a local system, there will exist some $x \in X$ such that $i_x^* H^i(\mathcal{F}) \neq 0$. This proves the equivalence above.

Next for $\mathcal{F} \in D_c^b(X)$, observe

$$\text{supp} H^j(\mathbb{D}\mathcal{F}) = \{x \in X \mid H^{-j}(i_x^! \mathcal{F}) \neq 0\}$$

This can be obtained by Verdier duality

$$i_x^! \mathcal{F} \simeq i_x^! \mathbb{D} \mathbb{D} \mathcal{F} \simeq \mathbb{D} i_x^* \mathbb{D} \mathcal{F}$$

Note that on $\{x\}$, the Verdier duality functor $\mathbb{D}_{\{x\}}$ is just the usual functor $\text{Hom}(-, k)$. So above equivalences compute

$$H^{-j}(i_x^! \mathcal{F}) \simeq H^j(\mathbb{D} \mathcal{F})_x^\vee$$

Finally for $\mathcal{F} \in D_{\text{loc}}^b(X)$, Lemma 0.7 in Talk 5 or [1, Thm. 2.2.13] yields $i_x^! \mathcal{F} \simeq i_x^* \mathcal{F}[-2n]$. So if $\mathcal{F} \in D_{\text{locf}}^{\geq -n}(X)$, $i_x^! \mathcal{F} \in D^{\geq n}(\text{Vect}_k)$ for all $x \in X$. In this case for $-j \geq n$ such that $H^j(\mathbb{D} \mathcal{F})$ is a non-zero local system, we will have $\dim \text{supp} H^j(\mathbb{D} \mathcal{F}) = n \leq -j$ which means $\mathcal{F} \in {}^p D_{\text{locf}}^{\geq 0}(X)$.

On the other hand if $\mathcal{F} \in {}^p D_{\text{locf}}^{\geq 0}(X)$, we have

$$\dim\{x \in X \mid H^{-j}(i_x^! \mathcal{F}) \neq 0\} \leq -j$$

Because if the support is non-empty then it will be X and has dimension n , for some j such that $H^j(\mathcal{F}) \neq 0$, we see $H^j(i_x^* \mathcal{F}) = H^j(i_x^! \mathcal{F}[2n]) = H^{j+2n}(i_x^! \mathcal{F}) \neq 0$. So that $n \leq j + 2n \Rightarrow j \geq -n$. So that $\mathcal{F} \in D_{\text{locf}}^{\geq -n}(X)$. \square

Perverse Sheaves along Immersions The following lemma describes how perverse t -structures behave along open and closed embeddings.

Lemma 1.13. *Let $j : U \hookrightarrow X$ be an open embedding and $i : Z \hookrightarrow X$ be a closed embedding.*

- (1). $j^*({}^p D_c^{\leq 0}(X)) \subseteq {}^p D_c^{\leq 0}(U)$ and $j^*({}^p D_c^{\geq 0}(X)) \subseteq {}^p D_c^{\geq 0}(U)$
- (2). $j_!({}^p D_c^{\leq 0}(U)) \subseteq {}^p D_c^{\leq 0}(X)$
- (3). $j_*({}^p D_c^{\geq 0}(U)) \subseteq {}^p D_c^{\geq 0}(X)$
- (4). $i_*({}^p D_c^{\leq 0}(Z)) \subseteq {}^p D_c^{\leq 0}(X)$ and $i^*({}^p D_c^{\geq 0}(Z)) \subseteq {}^p D_c^{\geq 0}(X)$
- (5). $i^*({}^p D_c^{\leq 0}(X)) \subseteq {}^p D_c^{\leq 0}(Z)$
- (6). $i^!({}^p D_c^{\geq 0}(X)) \subseteq {}^p D_c^{\geq 0}(Z)$

Proof. We prove this lemma using Verdier duality discussed in Talk 8.

At first for (1), notice that the restriction functor i.e. j^* is t -exact for the usual t -structure and will not increase the dimension of the support. This means for $\mathcal{F} \in {}^p D_c^{\leq 0}(X)$,

$$\dim \text{supp} H^i(j^* \mathcal{F}) = \dim \text{supp} j^* H^i(\mathcal{F}) \leq \dim \text{supp} H^i(\mathcal{F}) \leq -i$$

so $j^* \mathcal{F} \in {}^p D_c^{\leq 0}(U)$. Next since $j^! \simeq j^*$ for an open immersion j , j^* will commute with \mathbb{D} . Therefore for $\mathcal{G} \in {}^p D_c^{\geq 0}(X)$,

$$\dim \text{supp} H^i(\mathbb{D}(j^* \mathcal{G})) = \dim \text{supp} H^i(j^*(\mathbb{D} \mathcal{G})) \leq \dim \text{supp} H^i(\mathbb{D} \mathcal{G}) \leq -i$$

which means $j^* \mathcal{G} \in {}^p D_c^{\geq 0}(U)$.

For (2), the argument is similar to the above for j^* , because $j_!$ is t -exact as well and will not increase the dimension of the support by definition (see [1, Lem. 1.3.1]).

For (3), let $\mathcal{F} \in {}^p D_c^{\geq 0}(U)$. By Verdier duality, we have $\mathbb{D}(j_* \mathcal{F}) \simeq j_!(\mathbb{D} \mathcal{F})$. And $\mathbb{D} \mathcal{F} \in {}^p D_c^{\leq 0}(U)$ implies $j_!(\mathbb{D} \mathcal{F}) \in {}^p D_c^{\leq 0}(X)$ by part (2).

For (4), since i is a closed embedding, $i_* \simeq i_!$, the argument is the same as part (1). It says i_* is t -exact, will not increase the dimension of the support and commutes with \mathbb{D} .

For (5), it's similar to (2).

For (6), it's similar to (3). \square

The next lemma says a perverse sheaf is “locally” a shifted local system.

Lemma 1.14. *Let X be a variety and $(X_s)_{s \in \mathcal{I}}$ be a stratification on X . For every $s \in \mathcal{I}$, $j_s : X_s \rightarrow X$ is the inclusion map. Let $\mathcal{F} \in D_c^b(X)$.*

- (1). *Suppose \mathcal{F} is constructible with respect to \mathcal{I} . We have $\mathcal{F} \in {}^pD_c^{\leq 0}(X)$ if and only if $j_s^* \mathcal{F} \in D_{\text{loct}}^{\leq -\dim X_s}(X_s)$ for all $s \in \mathcal{I}$.*
- (2). *Suppose $\mathbb{D}\mathcal{F}$ is constructible with respect to \mathcal{I} . We have $\mathcal{F} \in {}^pD_c^{\geq 0}(X)$ if and only if $j_s^! \mathcal{F} \in D_{\text{loct}}^{\geq -\dim X_s}(X_s)$ for all $s \in \mathcal{I}$.*

Cohomology of Perverse Sheaves Let’s look at cohomologies of a perverse sheaf $\mathcal{F} \in \text{Perv}(X)$ where X has dimension n . The fact $\mathcal{F} \in {}^pD_c^{\leq 0}(X)$ obviously implies $H^i(\mathcal{F}) = 0$ for all $i > 0$. Next since $\mathcal{F} \in {}^pD_c^{\geq 0}(X)$, for all $\mathcal{G} \in {}^pD_c^{\leq -1}(X)$ we will have $\text{Hom}_{D_c^b(X)}(\mathcal{G}, \mathcal{F}) = 0^2$. Note that $D_c^{\leq -n}(X) \subseteq {}^pD_c^{\leq 0}(X)$ and therefore $D_c^{\leq -n-1}(X) \subseteq {}^pD_c^{\leq -1}(X)$. Then for all $\mathcal{G} \in D_c^{\leq -n-1}(X)$, $\text{Hom}_{D_c^b(X)}(\mathcal{G}, \mathcal{F}) = 0$. From [1, Lem. A.7.3] this means $\mathcal{F} \in D_c^{\geq -n}(X)$. So for all $i < -n$, $H^i(\mathcal{F}) = 0$.

As for $-n \leq i \leq 0$, $\mathcal{F} \in {}^pD_c^{\leq 0}(X)$ means $\dim \text{supp } H^i(\mathcal{F}) \leq -i$. Therefore $H^0(\mathcal{F})$ is supported at most on points, $H^{-1}(\mathcal{F})$ on curves, $H^{-n+1}(\mathcal{F})$ on divisors and $H^{-n}(\mathcal{F})$ everywhere.

Example 1.15. Let $\mathbb{C} = \mathbb{C}^\times \amalg \{0\}$ be a stratification for \mathbb{C} with inclusions $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$. In the Talk 2 (Example 1.27), we have computed the cohomology of $j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}$, whose stable of stalks are as follows.

	$\{0\}$	\mathbb{C}^\times
1	$\underline{\mathbb{C}}$	
0	$\underline{\mathbb{C}}$	$\underline{\mathbb{C}}$

So for $j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$,

	$\{0\}$	\mathbb{C}^\times
0	$\underline{\mathbb{C}}$	
-1	$\underline{\mathbb{C}}$	$\underline{\mathbb{C}}$

its 0-th cohomology is supported on the point 0 while its (-1) -th cohomology is supported everywhere. This diagram justifies the “lower triangularity” condition of perversity. Informally this means it will be a perverse sheaf. To see it formally, firstly it’s clear the above computations imply $j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD_c^{\leq 0}(\mathbb{C})$ directly. Next since $\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ is a perverse sheaf by Proposition 1.12, $j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD_c^{\geq 0}(\mathbb{C})$ by (3) of Lemma 1.13.

Actually we can also compute $\mathbb{D}j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ concretely. By Verdier duality we have

$$\begin{aligned}
 \mathbb{D}j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1] &= j_! \mathbb{D}\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \\
 &= j_! \mathbb{R}\mathcal{H}\text{om}(\underline{\mathbb{C}}_{\mathbb{C}^\times}[1], \underline{\mathbb{C}}_{\mathbb{C}^\times}[2]), \text{ since } \omega_{\mathbb{C}^\times} = \underline{\mathbb{C}}_{\mathbb{C}^\times}[2] \\
 &= j_! \mathcal{H}\text{om}(\underline{\mathbb{C}}_{\mathbb{C}^\times}[1], \underline{\mathbb{C}}_{\mathbb{C}^\times}[2]) \\
 &= j_! \underline{\mathbb{C}}_{\mathbb{C}^\times}[1]
 \end{aligned}$$

Note that by definition $j_! \underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ will vanish at 0. So its table of stalks is

	$\{0\}$	\mathbb{C}^\times
0		
-1		$\underline{\mathbb{C}}$

Then $\mathbb{D}j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD_c^{\leq 0}(\mathbb{C}) \Rightarrow j_* \underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD_c^{\geq 0}(\mathbb{C})$.

²The second condition of t -structure

References

- [1] Pramod N. Achar. *Perverse Sheaves and Applications to Representation Theory*, volume 258. American Mathematical Soc., 2021.