Higher Stacks and Derived Critical Loci

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Words are the boundary of thoughts.

Abstract

This Bachelor's thesis consists of three parts. In the first part we give a concise introduction to Toën and Vezzosi's work on homotopical algebraic geometry, a particular example of which is the derived algebraic geometry. Our main objects to study in this part are simplicial presheaves and we will show higher stacks are simplicial presheaves satisfying the descent for hypercovers. From this we can also obtain the concept of derived stacks. In the second part we focus on derived critical loci which is actually a derived intersection. We will show this construction is dual to BV-formalization in mathematical physics and serves a model for -1-shifted symplectic structure. Finally there is an appendix about homotopical algebra necessary for reading this thesis especially the first part. In this appendix we will talk about abstract homotopy theory in detail especially homotopy (co)limits and simplicial model categories.

Keywords: simplicial presheaves, higher stacks, derived critical loci

Contents

1 Introduction

In this introduction we first talk about some historical information or motivation for higher categories and higher stacks. After that we focus on what this thesis is about.

1.1 Towards Higher World

In mathematics the process that a concept is put forward by a mathematician is just like a pregnant woman gives birth to a child. Sometimes the birth goes well but sometimes you may need help from others or even worse a C-section is necessary. Moreover with time going, children will look really different from before. It's also true in mathematics that not all concepts are clear at the outset and with the further development for some concepts you can not even know their original appearance. However, it's fortunate that as long as you can conceive of a concept, no matter how vague it may be, we can express it clearly, even though this process of clarifying it may be long and painful. It's just the philosophy of early Wittgenstein. Most people know his famous proposition "What we cannot speak about we must pass over in silence" [74, 7] but I think another one is more important "Everything that can be thought at all can be thought clearly. Everything that can be put in words can be put clearly." [74, 4.116] As a corollary we can say **words are the boundary of thoughts**. The historical development of higher categories and higher stacks has proven this point. In the early history higher objects are like ghosts that we are aware of their existence but their true figure is so vague that we cannot look at them directly. But now we have had good theories to describe them.

There are many motivations for higher categories and higher stacks but here we begin with the problem of *non-abelian cohomology* which is also my first encounter with the higher world during my undergraduate study. Our main references here are [10], [38] and [61].

1.1.1 Non-abelian Cohomology

Our discussion here starts from the first order cohomology and then we introduce Giraud's method to define the second order non-abelian cohomology groups using *gerbes with liens*. Since the method using *torsors* to deal with the first order case comes from algebraic topology, we talk about (co)homology theory in algebraic topology first.

(Co)homology in Algebraic Topology A typical homology theory in algebraic topology is the *singular homology*. Suppose $|\Delta^n|$ is the geometric n-simplex and a *singular n-simplex* in a topological space X is a continuous map $\sigma : |\Delta^n| \to X$. $S_n(X)$ is the free abelian group whose basis consists of all singular n-simplexes. Then we can define the differential map

$$d_n : S_n(X) \to S_{n-1}(X), \;\; \sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma \epsilon_i^n$$

where $\epsilon_i^n : |\Delta^{n-1}| \to |\Delta^n|$ sends $(t_0, \cdots, t_{n-1})$ to $(t_0, \cdots, 0, t_i, \cdots, t_{n-1})$. This defines a singular chain complex $S(X)$ for any topological space X. Singular homology groups are defined to be homology groups of this chain complex. There are also some other homology theories such as simplicial homology. They are isomorphic to each other for some typical spaces and all satisfy Eilenberg-Steenrod axioms.

For any abelian group A we have (co)chain complexes with the coefficient A i.e. $\mathrm{Hom}(S(X), A)$ and $S(X)\otimes A$. (Co)homology groups $H^*(X; A)$ and $H_*(X; A)$ of X with coefficient A are defined to be (co)homology groups of these two new complexes.

Classifying Spaces Given an abelian group A for any non-negative integer n there exists an Eilenberg-Maclane space $K(A, n)$ such that its homotopy groups are trivial except $\pi_n(K(A, n)) = A$. We sketch how we can obtain such classifying spaces here. For $n \leq 1$, $K(G, n)$ exists for any group G which may not be abelian. When $n = 0$ we can just regard G as a topological space with the discrete topology. For $n = 1$

viewing G as a category with only one object whose morphism set is just the group G, then the geometric realization of the nerve $N(G)$ is homotopically equivalent to $K(G,1)$. This is a special case of the following Quillen equivalence

$$|\cdot|: s\mathbf{Set} \rightleftarrows \mathbf{Top}: \mathrm{Sing}$$

Such argument is also valid for any groupoid G and in this case the fundamental groupoid of $|N(G)|$ will just be G. For $n \geq 2$ the existence of $K(A,n)$ comes from Dold-Kan correspondence

$$N: s\mathbf{Ab} \rightleftarrows \mathbf{Ch}_{\geq 0}(\mathbb{Z}): \Gamma$$

Since homotopy groups are abelian for $n \geq 2$, $K(G,n)$ does not exist for $n \geq 2$ if G is non-abelian.

For every topological space X the class of maps from X to $K(A,n)$ up to homotopy is denoted by $[X, K(A,n)]$. Then a famous result in algebraic topology tells us there is a natural isomorphism

$$[X, K(A,n)] \xrightarrow{\sim} H^n(X;A)$$

for good enough space X such as CW-complexes. So this gives a way to define the first order non-abelian cohomology group as $[X, K(G,1)]$ for any non-abelian group G. From this there exists another characterization for $H^1(X;G)$, which is classified by *principal bundles*.

$p: E \to X$ is a *principal G-bundle* or *G-torsor* over X where G is a topological group if $p(g \cdot e) = p(e)$ and the induced map

$$G \times E \to E \times_X E, \ \ (g,e) \mapsto (g \cdot e, e)$$

is a homeomorphism. We use the symbol BG to mean the classifying space $K(G,1)$ for G. Then Milnor has proved there is a universal covering space EG over BG such that any principal G-bundle E on a CW-complex X can be obtained uniquely by a map $X \to BG$ up to homotopy, along which the pullback of EG will be isomorphic to E. In this sense the functor $\mathcal{P}_G$ of principal G-bundles up to isomorphism is represented by BG in the homotopy category of CW-complexes. Therefore the first non-abelian cohomology $H^1(X;G)$ also means the isomorphism class of principal G-bundles.

This interpretation is so powerful that we can define the first non-abelian cohomology for a topos which consists of internal G-torsors.

Torsors in a Grothendieck Topos Now we suppose $\mathcal{C}$ is a site with a Grothendieck pretopology and we consider the Grothendieck topos $Shv(\mathcal{C})$. Given a group sheaf $\mathcal{G}$ especially when $\mathcal{G}$ is abelian, we define its Čech complex for a cover $\{U_i \to X | i \in I\}$ as follows

$$0 \longrightarrow \prod_i \mathcal{G}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{G}(U_i \times_X U_j) \longrightarrow \cdots$$

where any element $(s_{i_1 \cdots i_n}) \in \coprod_{i_1, \cdots, i_n \in I} \mathcal{G}(U_{i_1 \cdots i_n})$ is sent to $s_{i_1 \cdots i_n}^{(-1)^{n+1}} \cdots s_{i_2 \cdots i_{n+1}}^{(-1)^1}$ at the position $\mathcal{G}(U_{i_1 \ldots i_{n+1}})$ where $U_{i_1 \cdots i_n} = U_{i_1} \times_X \cdots \times_X U_{i_n}$. In the abelian case it actually defines a complex. For non-abelian case we only focus on degree 1. Then in this case we have the group of cocyles

$$Z^1(\{U_i\}, \mathcal{G}) = \left\{ s_{ij} \in \prod_{i,j \in I} \mathcal{G}(U_i \times_X U_i) \middle| s_{ij} \cdot s_{jk} = s_{ik} \right\}$$

The equivalence relation is defined such that $(s_{ij}) \sim (t_{ij})$ if there exists some $(g_i) \in \prod_{i \in I} \mathcal{G}(U_i)$ satisfying $s_{ij} = g_i \cdot t_{ij} \cdot g_j^{-1}$. Then this gives a cohomology group $\check{H}^1(\{U_i\}, \mathcal{G})$. Considering all coverings for X, we obtain $\check{H}^1(X, \mathcal{G}) = \mathrm{colim} \check{H}(\{U_i\}, \mathcal{G})$. It's well known that this first order Čech cohomology group is actually classified by $\mathcal{G}$-torsors.

Definition 1.1.1. A sheaf $\mathcal{F}$ of sets is a *$\mathcal{G}$-torsor* if $\mathcal{G}$ acts on the left of $\mathcal{F}$ such that

(1). for every object X in $\mathcal{C}$, there is a cover $\{U_i \to X | i \in I\}$ such that $\mathcal{F}(U_i) \neq \emptyset$;

(2). $\mathcal{G} \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}, (g, x) \to (gx, x)$ is an isomorphism.

We use the symbol $\mathrm{Tors}(\mathcal{G}|X)$ to denote the category of torsors on the site $\mathcal{C}/X$. This is a groupoid. Actually for any $\mathcal{G}$-torsor $\mathcal{F}$, if $\mathcal{F}(U) \neq \emptyset$ then $\mathcal{F}|U \cong \mathcal{G}|U$. From this we have a criterion for $\mathcal{G}$-torsors.

Lemma 1.1.2. *If $\mathcal{F}$ is a $\mathcal{G}$-sheaf, then $\mathcal{F}$ is a $\mathcal{G}$-torsor if and only if for all $X \in \mathrm{Ob}(\mathcal{C})$, there is a cover $\{U_i \to X | i \in I\}$ such that $\mathcal{F}|U_i \cong \mathcal{G}|U_i$ as $\mathcal{G}$-sheaves.*

Now we give a proof how $\check{H}^1(X, \mathcal{G})$ classifies $\mathcal{G}$-torsors.

Theorem 1.1.3. *There is an isomorphism* $\mathrm{Tors}(\mathcal{G}|X)/\mathrm{iso} \cong \check{H}^1(X, \mathcal{G})$

Proof. At first from the first axiom of torsors, there is a cover $\{U_i \to X | i \in I\}$ for X such that $\mathcal{F}(U_i) \neq \emptyset$. Hence we can choose $s_i \in \mathcal{F}(U_i)$. Then from the second axiom of torsors, we know there exists $\exists g_{ij} \in \mathcal{G}(U_{ij}), g_{ij}s_j = s_i$ such that $g_{ij}s_j = s_i$. Note that this equality is valid in $\mathcal{F}(U_{ij})$. Since g_{ij} is unique, $g_{ik} = g_{ij}g_{jk}$ which means $(g_{ij}) \in Z^1(\{U_i\}, \mathcal{G})$. This gives an element in $\check{H}^1(X, \mathcal{G})$ which is independent from the choice of (s_i). If we choose another elemetns (t_i), from the second axiom of torsors we know $\exists! f_i \in \mathcal{G}(U_i),\ f_i s_i = t_i$. If $h_{ij}t_j = t_i$, then we have

$$h_{ij}f_j s_j = h_{ij}t_j = t_i = f_i s_i = f_i g_{ij} s_j$$

so that $h_{ij} = h = f_i g_{ij} f_j^{-1}$ which means $(h_{ij}) \sim (g_{ij})$.

On the other hand, an element in $\check{H}^1(X, \mathcal{G})$ can be represented by an element in $\check{H}^1(\{U_i\}, \mathcal{G})$. Hence we suppose $[(g_{ij})] \in \check{H}^1(\{U_i\}, \mathcal{G})$ and we can glue $\mathcal{G}|U_i = \mathcal{G}_{U_i}$ to obtain a $\mathcal{G}$-torsor. The gluing process is obtained by $g_{ij} : \mathcal{G}_{U_j}|U_{ij} \xrightarrow{\sim} \mathcal{G}_{U_i}|U_{ij}$. If we use an element in $\check{H}^1(\{V_j\}, \mathcal{G})$ to represent the cohomology class, then we can consider the refinement of the two coverings so that we can obtain isomorphic $\mathcal{G}$-torsors. □

Gerbes with Liens With the same idea we can define the second order non-abelian cohomology groups which also classifies some certain geometric objects. This problem is originally solved by Giraud in [22].

Definition 1.1.4. A *gerbe* on a site $\mathcal{C}$ is a stack $\mathcal{G}$ fibered in groupoids which is *locally non-empty* and *locally connected*. The two properties mean

- (locally non-empty): for any object X of $\mathcal{C}$, there is a covering $\{U_i \to |i \in X\}$ such that $\mathcal{G}(U_i)$ is non-empty;
- (locally connected): for any abject X of $\mathcal{C}$ and any two objects $a,\ b$ in $\mathcal{G}(X)$, there is a covering $\{U_i \to |i \in X\}$ such that there is an isomorphism $a|U_i \xrightarrow{\sim} b|U_i$ in every $\mathcal{G}(U_i)$.

Gerbes are classified by second non-abelian cohomology which is carefully defined by Giraud. For any object x in $\mathcal{G}(X)$, $\underline{\mathrm{Hom}}(x, x) = \underline{\mathrm{Aut}}(x)$ is a sheaf of automorphisms on $\mathcal{C}/X$. The sheaf condition follows from the descent property of stacks. From the first axiom of gerbes, there is a covering $\{U_i \to X | i \in X\}$ such that $\mathcal{G}(U_i)$ is non-empty. We choose an object x_i in every $\mathcal{G}(U_i)$ and $\underline{\mathrm{Aut}}(x_i)$ is a sheaf of groups on $\mathcal{C}/U_i$. Note that here $\mathcal{G}(X)$ may be empty. Then for $x_i|U_{ij}, x_j|U_{ij} \in \mathcal{G}(U_{ij})$, from the second axiom of gerbes there is covering $\{U_{ij}^{\xi} \to U_{ij} | \xi\}$ for U_{ij} and an isomorphism $f_{ij}^{\xi} : x_j|U_{ij}^{\xi} \xrightarrow{\sim} x_i|U_{ij}^{\xi}$. Then this induces an outer isomorphism of sheaves

$$\lambda_{ij}^{\xi} : \underline{\mathrm{Aut}}(x_j)|U_{ij}^{\xi} \to \underline{\mathrm{Aut}}(x_i)|U_{ij}^{\xi},\ \ u \mapsto f_{ij}^{\xi} \cdot u \cdot (f_{ij}^{\xi})^{-1}$$

where λ_{ij}^{ξ} is dependent on the choice of f_{ij}^{ξ} but for different choice λ_{ij}^{ξ}, λ_{ij}^{ζ} are equivalent on $U_{ij}^{\xi\zeta} = U_{ij}^{\xi} \times_{U_{ij}} U_{ij}^{\zeta}$. It's not difficult to see this. On $U_{ij}^{\xi\zeta}$, $\lambda_{ij}^{\xi}(u) = f_{ij}^{\xi} u (f_{ij}^{\xi})^{-1},\ \lambda_{ij}^{\zeta}(u) = f_{ij}^{\zeta} u (f_{ij}^{\zeta})^{-1}$. Suppose $f^{\xi\zeta} = f_{ij}^{\xi}(f_{ij}^{\zeta})^{-1} \in \underline{\mathrm{Aut}}(x_i)(U_{ij}^{\xi\zeta})$ and then $f^{\xi\zeta}\lambda_{ij}^{\zeta}(f_{ij}^{\xi\zeta})^{-1} = \lambda_{ij}^{\xi}$.

For any two sheaves $\mathcal{F}, \mathcal{H}$ of groups there is a concept of *sheaf of outer isomorphisms* $\underline{\mathrm{Out}}(\mathcal{F}, \mathcal{H})$ which is the associated sheaf of the presheaf

$$X \mapsto \mathrm{Iso}(\mathcal{F}|X, \mathcal{H}|X)/\mathcal{H}(X)$$

where $f \sim h \cdot f \cdot h^{-1}$ for $h \in \mathcal{H}(X)$. Any two outer isomorphisms can be composed. Given two outer isomorphisms $\mathcal{F} \xrightarrow{f} \mathcal{H} \xrightarrow{g} \mathcal{K}$ we have

$$(kgk^{-1}) \circ (hfh^{-1})(x) = (kgk^{-1})(hf(x)h^{-1}) = kg(h)(gf(x))(kg(h))^{-1}$$

and then $(kgk^{-1}) \circ (hfh^{-1}) \sim g \circ f$.

Now on a gerbe $\mathcal{G}$, for any two objects $x, \ y \in \mathcal{G}(X)$ we let $\underline{\mathrm{Out}}(x, y) := \underline{\mathrm{Out}}(\underline{\mathrm{Aut}}(x), \underline{\mathrm{Aut}}(y))$. Then $\{\lambda_{ij}^{\xi}\}$ given before can be glued to be a global outer isomorphism $\lambda_{ij} : \underline{\mathrm{Aut}}(x_j)|U_{ij} \to \underline{\mathrm{Aut}}(x_i)|U_{ij}$. A *lien* for a gerbe $\mathcal{G}$ is defined to be $\mathrm{lien}(\mathcal{G}) := (\underline{\mathrm{Aut}}(x_i), \lambda_{ij})$ where $\lambda_{ii} = \mathrm{id}$. $\lambda_{ij} \circ \lambda_{jk}$ locally sends u to $f_{ij}f_{jk}uf_{jk}^{-1}f_{ij}^{-1} = (f_{ij}f_{jk})u(f_{ij}f_{jk})^{-1}$. Therefore $\lambda_{ij} \circ \lambda_{jk}$ are equivalent to λ_{ik} as outer isomorphisms.

Definition 1.1.5 (Lien)**.** An abstract *lien* $\mathcal{F}$ on $\mathcal{C}/X$ represented by a covering $\mathcal{U} = \{U_i \to X | i \in I\}$ is a collection of sheaves of groups $\mathcal{F}_i$ on $\mathcal{C}/U_i$ and outer isomorphisms $\lambda_{ij} : \mathcal{F}_j|U_{ij} \to \mathcal{F}_i|U_{ij}$ such that $\lambda_{ii} = \mathrm{id}$ and $\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik}$.Given another abstract lien $\mathcal{H} = (\mathcal{H}_k, \mu_{kl})$ represented by another cover $\mathcal{V}$, an isomorphism between liens consists of a common refinement $\{W_\alpha \to X | \alpha \in A\}$ of $\mathcal{U}$ and $\mathcal{V}$, and outer isomorphisms $\varphi_\alpha : \mathcal{F}_\alpha \to \mathcal{H}_\alpha$ such that $\varphi_\alpha \lambda_{\alpha\beta} = \mu_{\alpha\beta}\varphi_\beta$.

Hence for a gerbe $\mathcal{G}$ different coverings and different choices of elements will define isomorphic liens. Then a gerbe $\mathcal{G}$ with lien $\mathcal{F}$ consists of $(\mathcal{G}, \theta)$ where $\theta : \mathrm{lien}(\mathcal{G}) \xrightarrow{\sim} \mathcal{F}$. All gerbes with the lien $\mathcal{F}$ on $\mathcal{C}/X$ up to isomorphism is denoted by $\mathrm{Gerbes}(X, \mathcal{F})$. Then a famous result in [22] asserts gerbes with the line $\mathcal{F}$ is classified by the second cohomology group of $\mathcal{F}$.

Theorem 1.1.6. $\mathrm{Gerbes}(X, \mathcal{F}) \cong \check{H}^2(X, \mathcal{F})$

The second cohomology group for a lien is given by really complicated relationships between 2-cocycles. Details can be found in [10, Section 2] and [22].

Higher Case Now there is a natural question what higher non-abelian cohomollogy theories are like. An essential point is that higher non-abelian cohomology must classify some certain higher geometric objects. In lower cases they classify torsors and gerbes which are sheaves and stacks respectively. So a natural hypothesis is that nth cohomology theory classifies some certain n-stacks. [10] tried to solve this problem for 2-stacks and the third non-abelian cohomology theory.

In [26] Grothendieck gives his great insight about higher non-abelian cohomology theory. For an object X in a site $\mathcal{C}$ the cohomology of X with coefficient an n-stack $\mathcal{F}$ should be the n-category n-$\mathrm{Stacks}_{\mathcal{C}/X}(X, \mathcal{F})$ of global sections of $\mathcal{F}$. With this idea in [38] Lurie defines the cohomology theory for any ∞-topos [38, Definition 7.2.2.14] and the $(n+1)$th cohomology theory just classifies what he calls n-gerbes. [38] gives a detailed answer to higher cohomology theories and has obtained a great success.

1.1.2 Higher Stacks

We know roughly speaking classical stacks are (pseudo)functors from a site to **Gpd** the category of groupoids. So intuitively higher stacks should be certain functors from a site to higher groupoids. Just like classical stacks satisfying the descent condition, higher stacks should also satisfy some certain higher descent condition. Now there are at lest two problems. What are higher groupoids? What's the higher descent condition?

Homotopy Hypothesis As we have talked, for any groupoid there is a classifying space whose fundamental groupoid is just this groupoid. Therefore in this sense we have an equality

groupoid=homotopy 1-type

Then a natural hypothesis should be

n-groupoid=homotopy n-type

Especially when n converges to infinity, ∞-groupoids should be equivalent to homotopy ∞-types which roughly mean topological spaces. So from this hypothesis, an ∞-groupoid should at least contain all information of homotopy groups of a topological space. This is *homotopy hypothesis* which means any theory for ∞-groupoids should at least induce an equivalence between ∞-groupoids and the localization of topological spaces with respect to weak homotopy equivalences. This is a test for theories of ∞-groupoids.

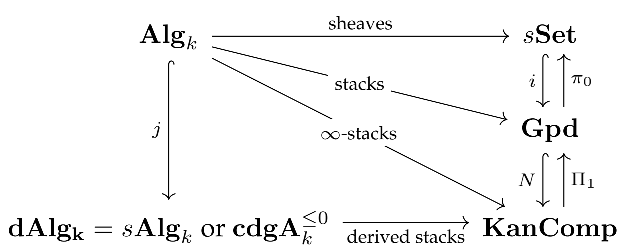
In [26] Grothendieck gives really complicated axioms for ∞-groupoids where he tries to describe morphisms in an ∞-groupoid directly and proves such defined ∞-groupoids satisfy the homotopy hypothesis. But nowadays we know Kan complexes serve a good model for ∞-groupoids [14, Theorem 3.5.1] and the Quillen equivalence between simplicial sets and topological spaces tells us that Kam complexes really satisfy the homotopy hypothesis. From this point it seems Grothendeick's effort in [26] is in vain. However, it's not really true. In Grothendieck's opinion it's tautological to define an ∞-groupoid as a Kan complex and in a letter to Tim Porter, he gave some reasons why he disliked this definition

- simplicial sets are not globular like the intuition of higher categories;
- composition in a Kan complex is not defined precisely, only up to homotopy.

But nowadays we just suppose the homotopy hypothesis holds and define ∞-groupoids as Kan complexes. In this thesis we also take this point. As for Grothendieck's direct definition for ∞-groupoids, you can find details in [42].

Derived Stacks From our statements above, an ∞-prestack should be a functor from a site to the category of ∞-groupoids i.e. Kan complexes. This actually generalizes the classical case since via the nerve functor a usual groupoid is just a Kan complex whose higher homotopy groups are trivial. As for the descent condition, it should be in homtopy sense. Here we take the point in [16] and say an ∞-prestack is an ∞-stack if it satisfies the descent condition in [16] (Definition 2.1.9).

If we let the site $\mathcal{C}$ be $\mathbf{Aff}_k$ the category of affine schemes, then our definition for higher stacks generalizes the classical stacks appearing in moduli theory. Since affine schemes are dual to commutative algebras, if we consider the opposite category $\mathbf{dAff}_k$ of some certain derived commutative algebras such as simplicial commutative algebras or commutative differential graded algebras, then we will obtain the concept of derived prestakcs and derived stacks. Theses concepts can be summarized in the following diagram due to Toën and Vezzosi.



For the definition of derived stacks, the first problem we should deal with is how we can equip derived commutative algebras with a Grothendieck topology so that we can define the descent condition. There is a concept *model pre-topology* (Definition 2.2.5) in [67] which can be defined in a model category so that it induces a Grothendieck topology on the homotopy theory of this model category. With this concept we can define higher stacks on a *model site* and derived stacks are a concrete example of these higher stacks.

1.2 About this Thesis

This thesis is supervised by Prof. Joost Nuiten during my stay in France and it's written as a Bachelor's thesis for Shandong University and an M1 thesis for Université Toulouse III. When I was a junior in Shandong University, it was my first time to get to know something about simplicial theory or more precisely abstract homotopy theory. At that time I have learned some Grothendieck's formalization for algebraic geometry i.e. scheme theory especially his functorial point in FGA and the new edition of EGA I. This point is emphasized in a reply of Prof. Bosch sending to me. He is really an enthusiastic professor and in his letter he advised me to learn the Hilbert scheme which is really important no matter for this object itself or the philosophy behind it. Since then I have been very curious about whether there are some connections between what I have learned i.e. abstract homotopy theory and functorial algebraic geometry. With this motivation Toën and Vezzosi's works on homotopical algebraic geometry really open a new door for me so I plan to go to France to complete my Bachelor's thesis in this area.

Apart from the introduction, this thesis consists of three parts. In the first part we give a concise but precise introduction to the subject homotopical algebraic geometry which is mainly the work of Toën and Vezzosi in [67] and [68]. In this part we talk about simplicial presheaves on a usual Grothendieck site and a model site which serve a concrete model for *higher stacks*. We also consider how we can equip the opposite category of commutative differential graded algebras with a model pretopology so that we can obtain the concept of *derived stacks*. In this sense this part is also an introduction to derived algebraic geometry.

The second part is about *derived critical loci*. For a real smooth manifold M of dimension n, we suppose f is its global section i.e. a smooth function $f : M \to \mathbb{R}$. Then in differential geometry a *critical point* is defined such that the induced function on tangent spaces at this point is not surjective. In our case here the differential df should be zero at that point. Therefore the *critical locus* is just $(df)^{-1}(0)$. In some cases such as f is Morse, the critical locus only consists of discrete points. But if we allow (M, f) to be more singular, the structure of critical loci will be more complicated.

There is also a more categorical way to see critical loci. It's just the following fiber product

$$\begin{array}{ccc} \mathrm{Crit}(f) & \longrightarrow & M \\ \downarrow & \lrcorner & \downarrow{\scriptstyle 0} \\ M & \xrightarrow[df]{} & \mathrm{T}^*M \end{array}$$

where T^*M is the cotangent bundle over M, 0 sends x in M to $(x, 0)$ and

$$df : M \to \mathbb{T}^*M, \;\; x \mapsto (x, df(x)) = (x, \frac{\partial f}{\partial x_1}(x)dx_1, \cdots, \frac{\partial f}{\partial x_1}(x)dx_n)$$

Since M is Hausdorff, the diagonal map $\Delta : M \to M \times M$ is a homeomorphism to the image $\Delta(M)$. So the fiber product $\mathrm{Crit}(f)$ is equivalent to $(df)^{-1}(0)$. It's well known that in general there do not exist fiber products in the category of smooth manifolds. Here $\mathrm{Crit}(f)$ will not be a smooth manifold. But in this thesis we consider the derived version of this construction in the context of algebraic geometry which is also valid in derived differential geometry. So for our purpose in this thesis, the critical loci should be in the derived sense and it's actually defined as a homotopy fiber product. In this second part following [70] we will show this construction is dual to *BV-formalization* and serves a model for -1-shifted symplectic structure.

The third part is an appendix about many concepts and techniques in homotopical algebra i.e. model categories which is necessary for reading this thesis especially the first part. In this appendix we give detailed discussions about abstract homotopy theory and nearly all proofs of theorems stated are given. Actually many constructions are based on homotopical algebra. For example the category of higher stacks is actually a *left Bousfield localization* of the projective model category of simplicial presheaves. And the derived critical locus is in fact a homotopy fiber product.

1.3 Acknowledgement

Firstly I want to thank Prof. Joost Nuiten for giving me the topic of derived critical loci which leads me to the world of derived symplectic geometry, and for his patient guidance and useful suggestions on not only the thesis but also the choice of master schools.

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Finally I want to thank Wittgenstein since his philosophy has helped me through some tough times and also provides me a different perspective on mathematics.

2 Homotopical Algebraic Geometry

In this section we give a concise introduction to the subject homotopical algebraic geometry, an important example of which is the derived algebriac geometry. Main references for this section are notes and papers of Bertrand Toën and Gabriele Vezzosi especially [67] and [68].

2.1 Simplicial Presheaves

Suppose $\mathcal{C}$ is a small category and then a *simplicial presheaf* is a simplicial object on $Pr(\mathcal{C}) = \mathbf{Set}^{\mathcal{C}^{op}}$ i.e. a functor $\Delta^{op} \to Pr(\mathcal{C})$ which is also equivalent to the functor $\mathcal{C}^{op} \to s\mathbf{Set}$. The category of simplicial presheves is denoted by $sPr(\mathcal{C})$. Since $s\mathbf{Set}$ is a combinatorial model category, $sPr(\mathcal{C})$ can be equipped with projective and injective model structures (Remark A.4.5). And moreover it's combinatorial as well.

Remark 2.1.1. We suppose $sPr(\mathcal{C})$ is with the projective model structure. Then from the proof of Theorem A.4.3 and the small object argument (Theorem A.1.11), cofibrations in $sPr(\mathcal{C})$ are especially objectwise cofibrations. And since pushouts in $sPr(\mathcal{C})$ are objectwise, objectwise weak equivalences are preserved by pushouts along cofibrations. Therefore $sPr(\mathcal{C})$ is left proper which comes from the left properness of $s\mathbf{Set}$. Actually $sPr(\mathcal{C})$ is proper.

Remark 2.1.2. $sPr(\mathcal{C})$ has a natural simplicial model structure. For any simplicial set X, we also use the symbol X to denote the constant simplcial presheaf $X : \mathcal{C}^{op} \to s\mathbf{Set}$. Then

$$\otimes : s\mathbf{Set} \times sPr(\mathcal{C}) \to sPr(\mathcal{C}), \quad (X, \mathcal{F}) \mapsto X \times \mathcal{F}$$

If the right adjoint functor

$$\mathrm{Map} : sPr(\mathcal{C})^{op} \times sPr(\mathcal{C}) \to s\mathbf{Set}$$

exists, then it must satisfy

$$\mathrm{Hom}_{sPr(\mathcal{C})}(X \times \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{s\mathbf{Set}}(X, \mathrm{Map}(\mathcal{F}, \mathcal{G}))$$

Now we let $X = \Delta^n$. Therefore we obtain

$$\mathrm{Hom}_{s\mathbf{Set}}(\Delta^n, \mathrm{Map}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Map}(\mathcal{F}, \mathcal{G})_n \cong \mathrm{Hom}_{sPr(\mathcal{C})}(\Delta^n \times \mathcal{F}, \mathcal{G})$$

Here we just define $\mathrm{Map}(\mathcal{F}, \mathcal{G})_n = \mathrm{Hom}_{sPr(\mathcal{C})}(\Delta^n \times \mathcal{F}, \mathcal{G})$. Proofs that it actually defines a functor and moreover is right adjoint to $\otimes$ are similar to the case of presheaves. For the case of presheaves, you can find details in [41, Section I.6].

Next to prove it's actually a simplicial model category, it's necessary to prove for a cofibration $K \rightarrowtail L$ in $s\mathbf{Set}$ and a fibration $\mathcal{F} \twoheadrightarrow \mathcal{G}$ in $sPr(\mathcal{C})$, $\mathcal{F}^L \to \mathcal{F}^K \times_{\mathcal{G}^K} \mathcal{G}^L$ will be a fibration in $sPr(\mathcal{C})$ which is a weak equivalence if either of domain maps is a weak equivalence. But in $sPr(\mathcal{C})$, weak equivalences and fibrations are defined objectwise, this statement just follows the simplicial model structure on $s\mathbf{Set}$. Note that $\mathcal{F}^K(c) = \mathcal{F}(c)^K$ is the *function complex* in $s\mathbf{Set}$ for every object $c \in \mathrm{Ob}(\mathcal{C})$.

This argument can be applied to any simplicial model category $\mathcal{M}$. Then $\mathcal{M}^{\mathcal{C}}$ will have the projective simplicial model structure.

Remarks above just say $sPr(\mathcal{C})_{proj}$ is a left proper combinatorial simplicial model category.

Remark 2.1.3 (Enriched Yoneda's Lemma)**.** For any object $c \in \mathrm{Ob}(\mathcal{C})$, we can view is as a simplicial presheaf such that for any $a \in \mathrm{Ob}(\mathcal{C})$, $c(a)$ is the constant simplicial set $\mathrm{Hom}_{\mathcal{C}}(a, c)$. This also means for any $[n]$ in Δ, c_n is just the representable functor $\mathcal{C}^{op} \to \mathbf{Set}$ of c. Since $[0]$ is the terminal object in Δ, there exists a unique map $[n] \to [0]$. Hence any morphism $c \to \mathcal{F}$ in $sPr(\mathcal{C})$ in completely determined by the morphism $c_0 \to \mathcal{F}_0$ in $Pr(\mathcal{C})$

$$\begin{array}{ccc} c_0 & = & c_n \\ \downarrow & & \downarrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_n \end{array}$$

Then we will have

$$\begin{aligned}
\mathrm{Map}(c,\mathcal{F})_n = \mathrm{Hom}_{s\mathbf{Set}}(\Delta^n, \mathrm{Map}(c,\mathcal{F})) &\cong \mathrm{Hom}_{sPr(\mathcal{C})}(\Delta^n \otimes c, \mathcal{F}) \cong \mathrm{Hom}_{sPr(\mathcal{C})}(c, \mathcal{F}^{\Delta^n}) \\
&= \mathrm{Hom}_{Pr(\mathcal{C})}(c_0, \mathcal{F}_0^{\Delta^n}) = \mathcal{F}_0^{\Delta^n}(c) \\
&= \mathcal{F}(c)_0^{\Delta^n} = \mathrm{Hom}_{s\mathbf{Set}}(\Delta^0, \mathcal{F}(c)^{\Delta^n}) \\
&\cong \mathrm{Hom}_{s\mathbf{Set}}(\Delta^0 \times \Delta^n, \mathcal{F}(c)) \\
&= \mathrm{Hom}_{s\mathbf{Set}}(\Delta^n, \mathcal{F}(c)) \cong \mathcal{F}(c)_n
\end{aligned}$$

Therefore $\mathrm{Map}(c,\mathcal{F}) = \mathcal{F}(c)$ in $s\mathbf{Set}$. It's the *enriched Yoneda's lemma*.

In the following we suppose $\mathcal{C}$ is a *site* i.e. a category with *Grothendieck topology* or *pretopology*. For definitions you can read [41, Section III.2].

Definition 2.1.4. A map $f : \mathcal{F} \to \mathcal{G}$ of simplicial presheaves is called a *local trivial fibration* if it has the RLP wrt $\partial\Delta^n \to \Delta^n$ locally for all $n \geq 0$, which means for any object c of $\mathcal{C}$and any lifting problem

$$\begin{array}{ccc}
\partial\Delta^n \otimes c & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
\Delta^n \otimes c & \longrightarrow & \mathcal{G}
\end{array}$$

there exists a covering sieve $\{u_i \to c | i \in I\}$ such that the lifting problem above admits a solution on this sieve in the sense that for any $u_i \to c$

$$\begin{array}{ccccc}
\partial\Delta^n \otimes u_i & \longrightarrow & \partial\Delta^n \otimes c & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^n \otimes u_i & \longrightarrow & \Delta^n \otimes c & \longrightarrow & \mathcal{G}
\end{array}$$

Remark 2.1.5. From Remark 2.1.3, $\mathrm{Map}(c,\mathcal{F}) = \mathcal{F}(c)$. Therefore the definition of locally trivial fibrations above also means for any object c of $\mathcal{C}$, there is a covering sieve $\{u_i \to c\}$ such that the following lifting problem solves

$$\begin{array}{ccccc}
\partial\Delta^n & \longrightarrow & \mathcal{F}(c) & \longrightarrow & \mathcal{F}(u_i) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \mathcal{G}(c) & \longrightarrow & \mathcal{G}(u_i)
\end{array}$$

Note that this is different from saying $\mathcal{F}(u_i) \to \mathcal{G}(u_i)$ is trivial fibration in $s\mathbf{Set}$.

Proposition 2.1.6. *A map $f : \mathcal{F} \to \mathcal{G}$ of simplicial presheaves has the RLP wrt a map $i : X \to Y$ of simplicial sets locally if and only if the induced map*

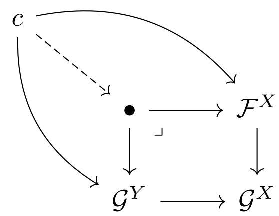
$$(i^*, f_*) = \widehat{\{i, f\}} : \mathcal{F}^Y \to \mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y$$

of simplicial presheves is a local epimorphism in degree 0 *as a morphism of presheaves in* $Pr(\mathcal{C})$.

Proof. Given a lifting problem

$$\begin{array}{ccc}
X \otimes c & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
Y \otimes c & \longrightarrow & \mathcal{G}
\end{array}$$

it has solution $Y \otimes c \to \mathcal{F}$ if and only if the induce map $c \to \mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y$ factors through $\mathcal{F}^Y$.



Since this problem has a solution locally, $c \to \mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y$ will factor through $\mathcal{F}^Y$ locally. From Remark 2.1.3, the map $c \to \mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y$ is totally determined by the morphism at zero level. But $c_0 \to (\mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y)_0$ is equivalent to an element in $(\mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y)_0(c)$. Therefore the locally lifting property actually means any element in $(\mathcal{F}^X \times_{\mathcal{G}^X} \mathcal{G}^Y)_0(c)$ admits a local lifting to $\mathcal{F}_0^Y$ which is equivalent to saying $\widehat{\{i, f\}}_0$ is locally surjective. □

So there is another characterization for local trivial fibrations, i.e. $\mathcal{F}^{\Delta^n} \to \mathcal{F}^{\partial\Delta^n} \times_{\mathcal{G}^{\partial\Delta^n}} \mathcal{G}^{\Delta^n}$ is a local epimorphism at the level of 0.

Definition 2.1.7. Let c be an object in $\mathcal{C}$. A *hypercover* of c consists of a map $\mathcal{U} \to c$ of simplicial presheves such that

(1). for each $n, \mathcal{U}_n$ is a product of repersentable presheaves on $\mathcal{C}$

(2). the map $\mathcal{U} \to c$ is a local trivial fibration.

Example 2.1.8. For any covering sieve $\{u_i \to c\}$, its corresponding *Čech nerve* $N(u)$ is defined to be the simplicial presheaf such that

$$N(u)_n := \coprod_{(i_0, \cdots, i_n)} u_{i_0} \times_c \cdots \times_c u_{i_n}$$

For any $(i_0, \cdots, i_n)$ there is a natual map $u_{i_0} \times_c \cdots \times_c u_{i_n} \to c$ and this induces a map $N(u) \to c$ of simplicial presheaves.

For any object a of $\mathcal{C}$ and any lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & N(u)(a) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & c(a) \end{array}$$

since $\Delta^n \to c(a)$ is just an element in $\mathrm{Hom}_{\mathcal{C}}(a, c)$, we can obtain a covering sieve on a along this map $a \to c$. Note that $\partial\Delta^n$ is the coequalizer of

$$\textstyle\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{0 \le k \le n} \Delta^{n-1} \longrightarrow \partial\Delta^n \subseteq \Delta^n$$

for $n \ge 2$. Then $\partial\Delta^n \to N(u)(a)$ consists of some elements in $N(u)(a)_{n-1}$ which coincide on $N(u)(a)_{n-2}$. This means for two maps $(f_0, \cdots, \hat{f}_i, \cdots, f_n)$ and $(g_0, \cdots, \hat{g}_j, \cdots, g_n)$ from a to $u_0 \times_c \cdots \times_c \hat{u}_i \times_c \cdots \times_c u_n$, where $f_k, g_k : a \to u_k$ and $\hat{\bullet}$ means this element does not exist,

$$(f_0, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots, f_n) = (g_0, \cdots, \hat{g}_i, \cdots, \hat{g}_j, \cdots, g_n)$$

If $n \ge 2$, obviously there will exist a lifting map from a to $u_0 \times_c \cdots \times_c u_n$ which is equivalent to the lifting $\Delta^n \to N(u)(a)$.

If $n = 1$, $\partial\Delta^1 = \Delta^0 \coprod \Delta^0$. Then $\partial\Delta^1 \to N(u)(a)$ consists of a pair of maps $(f : a \to u_0, g : a \to u_1)$ such that when composing $u_0 \to c$ and $u_1 \to c$, there is an element in $c(a)_1$ connecting them, which will mean (f, g) defines a map $a \to u_0 \times_c u_1$. It's just the lifting.

If $n = 0$, then $\partial\Delta^0 = \emptyset$. Though a map $a \to c$ may not be lifted to be a map $a \to u_i$ for some i, there can exist a covering sieve of a such that they factor through some u_i. This covering sieve can be just obtained by $\{u_i \to c\}$ along the map $a \to c$.

This example gives the intuition of a hypercover.

For a hypercover $\mathcal{U} \to c$, we can just write it as

$$\cdots \longrightarrow \coprod u_1^{i_1} \overrightarrow{\underleftarrow{\longrightarrow}} \coprod u_0^{i_0} \longrightarrow c$$

where $\mathcal{U}_n = \coprod_{i_n} u_n^{i_n}$

Definition 2.1.9. A fibrant object $\mathcal{F}$ in $sPr(\mathcal{C})_{proj}$ satisfies the *descent* for a hypercover $\mathcal{U} \to c$ if the natural map from $\mathcal{F}(c)$ to the homotopy limit of the diagram

$$\prod_{i_0} \mathcal{F}(u_0^{i_0}) \overrightarrow{\underleftarrow{\longrightarrow}} \prod_{i_1} \mathcal{F}(u_1^{i_1}) \longrightarrow \cdots$$

is a weak equivalence between simplicial sets. Here products range over representable summands of each $\mathcal{U}_n$. If $\mathcal{F}$ is not fibrant, we say it satisfies the *descent* if its fibrant replacement satisfies thie property.

Remark 2.1.10. The diagram in the definition above is obtained by applying the functor $\mathrm{Map}(-, \mathcal{F})$ to the diagram of hypercover $\mathcal{U} \to c$. We denote this diagram by $\mathcal{F}(\mathcal{U})$.

Remark 2.1.11. Though the collection of all hypercovers may be big, we can find a small set S of hypercovers which is *dense* in the sense that every hypercover can be obtained by refining a nice hypercover in this set [16, Proposition 6.6]. When $\mathcal{C}$ is better enough i.e. a *Verdier site* [16, Definition 9.1], there will exist such small set S, elements in which are cofibrant in $sPr(\mathcal{C})_{proj}$ [16, Theorem 9.6]. After having the small set S, we can do left Bousfield localization on $sPr(\mathcal{C})_{proj}$ to obtain the local model category $sPr(\mathcal{C})_{proj}^{loc}$. Similarly we also have $sPr(\mathcal{C})_{inj}^{loc}$ which is just the *Jardine's model structure* on $sPr(\mathcal{C})_{Jar}$. Details can be found in [16, Theorem 6.2]. In this way, weak equivalences in local model categories are just *local weak equivalences* which means they induce weak equivalences on stalks.

Remark 2.1.12 (Jardine's Model Structure)**.** On $sPr(\mathcal{C})_{Jar}$, weak equivalences are local weak equivalences. We need to explain the concept *local weak equivalence* [35, Section 4.1]. For a map $f : \mathcal{F} \to \mathcal{G}$ of simplicial presheaves can induce maps of presheaves. For $n = 0$, $\pi_0\mathcal{F}$ is a presheaf of path components of $\mathcal{F}$ such that for any object c of $\mathcal{C}$, $\pi_0\mathcal{F}(c) = \pi_0(\mathcal{F}(c))$. For $n > 0$ we can define a presheaf $\pi_n(\mathcal{F}|c, x) : (\mathcal{C} \downarrow c)^{op} \to \mathbf{Set}$ where $x \in \mathcal{F}_0(c)$ such that it sends any map $f : u \to c$ to the homotopy group $\pi_n(\mathcal{F}(u), f^*(x))$, where we can replace $\mathcal{F}(u)$ by any its fibrant replacement and define the homotopy group to be the *simplicial homotopy group* of that Kan complex or we can just let $\pi_n(\mathcal{F}(u), f^*(x)) = \pi_n(|\mathcal{F}(u)|, f^*(x))$ the homotopy group of the topological space $|\mathcal{F}(u)|$. Next we use $\tilde{\pi}_0\mathcal{F}$ and $\tilde{\pi}_n(\mathcal{F}|c, x)$ to denote sheaves associated with $\pi_0\mathcal{F}$ and $\pi_n(\mathcal{F}|c, x)$ respectively. A local weak equivalence means it induces isomorphisms on these homotopy sheaves.

If $\mathcal{C}$ is the category of open subsets of some topological space X, then this definition of local weak equivalences are equivalent to that they induce weak equivalences $\mathcal{F}_x \to \mathcal{G}_x$ in stalks for any point $x \in X$. This is also valid for sites having enough points.

Cofibrations in $sPr(\mathcal{C})_{Jar}$ are objectwise cofirbations and fibrations are those having the RLP wrt to all maps which are cofibrations and local weak equivalences. These make $sPr(\mathcal{C})_{Jar}$ a proper simplicial model category [35, Theorem 5.8]. In this way it's difficult to describe fibrant objects which is clear when focusing on $sPr(\mathcal{C})_{inj}^{loc}$.

In the following we prove local objects in local model categories are just those satisfying descents.

Lemma 2.1.13 (Co-Yoneda Lemma). [1] *Suppose $\mathcal{C}$ is a small category and $h : \mathcal{C} \to Pr(\mathcal{C})$ is the Yoneda embedding. Then any presheaf $P : \mathcal{C}^{op} \to$ **Set** is the coequalizer of the following diagram*

$$\coprod_{\substack{u:c'\to c\in\mathcal{C}\\ p\in P(c)}} h(c') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{\substack{c\in\mathcal{C}\\ p\in P(c)}} h(c) \xrightarrow{\ \epsilon\ } P$$

where for any object b of $\mathcal{C}$ and maps $v : b \to c$ or c',

$$\theta_{u,p}(v) = (c, p; uv),\ \tau_{u,p}(v) = (c', P_u(p); v) \text{ and } \epsilon_{c,p}(v) = P_v(p)$$

.

Proof. Since colimits and limits of presheaves are objectwise, we can let the object b fixed and prove $P(b)$ is the coequalizer.

At first we prove $\epsilon \circ \theta = \epsilon \circ \tau$. For $v : b \to c'$, $\theta_{u,p}(v) = (c, p; uv)$ and $\tau_{u,p}(v) = (c', P_u(p); v)$. Then $P_v(P_u(p)) = P_{uv}(p)$.

Next we give a map $\pi : \coprod_{c\in\mathcal{C},\ p\in P(c)} \mathrm{Hom}_{\mathcal{C}}(b, c) \to X$ such that $\pi \circ \theta = \pi \circ \tau$. Notice that $\epsilon_{b,p}(\mathrm{id}_b) = P_{\mathrm{id}_b}(p) = p$. ϵ is surjective and if there exits some $\delta : P(b) \to X$ satisfying $\delta \circ \epsilon = \pi$, then δ will be unique. To prove the existence of δ, we just need to prove for any two elements $(c_1, p_1; v_1)$ and $(c_2, p_2 : v_2)$ such that $\epsilon(c_1, p_1; v_1) = \epsilon(c_2, p_2; v_2)$ then $\pi(c_1, p_1; v_1) = \pi(c_2, p_2 : v_2)$. $\epsilon(c_1, p_1; v_1) = \epsilon(c_2, p_2 : v_2)$ means $P_{v_1}(p_1) = P_{v_2}(p_2)$ for $v_i : b \to c_i$.

Since $\theta_{v_i,p_i}(\mathrm{id}_b) = (c_i, p_i; v_i)$ and $\tau_{v_i,p_i}(\mathrm{id}_b) = (b, P_{v_i}(p_i); \mathrm{id}_b)$, $\tau_{v_1,p_1}(\mathrm{id}_b) = \tau_{v_2,p_2}(\mathrm{id}_b)$. Then $\pi_{c_i,p_i}(v_i) = \pi \circ \theta_{v_i,p_i}(\mathrm{id}_b) = \pi \circ \tau_{v_i,p_i}(\mathrm{id}_b) = \pi_{b,P_{u_i}(p_i)}(\mathrm{id}_b)$. □

Remark 2.1.14. There is a natural way for $Pr(\mathcal{C})$ to be enriched over **Set** such that $S \otimes P = \coprod_{s\in S} P \cong S \times P$ where P is a presheaf and S is a set viewing it as a constant presheaf as well. Then the above lemma actually says $P \cong \int^{c\in\mathcal{C}} P(c) \otimes h(c)$ is the *coend*. For a simplicial version you can look at the Definition A.5.17.

For a presheaf $P : \mathcal{C}^{op} \to$ **Set**, we can define a $\mathcal{C}^{op}$-indexed diagram D_P in $Pr(\mathcal{C})$ such that for any object c of $\mathcal{C}$, $D_P(c)$ is the constant presheaf P_c of the set $P(c)$. Then $\int^{c\in\mathcal{C}^{op}} h(c) \times D_P(c)$ is just $\int^{c\in\mathcal{C}} P(c) \otimes h(c)$ since $\coprod_{\substack{u:c'\to c\\ p\in P(c)}} h(c') \cong \coprod_{u:c'\to c} \big(h(c') \times P_c\big)$. Therefore $P \cong \int^{c\in\mathcal{C}^{op}} h(c) \times D_P(c)$.

Lemma 2.1.15. *If $\mathcal{F}$ is a simplicial presheaf and we define a Δ^{op}-indexed diagram in $sPr(\mathcal{C})$ such that it sends $[n]$ to $\mathcal{F}_n$ which is a presheaf of sets but we view it as a discrete simplicial presheaf. Then the geometric realization $|D_{\mathcal{F}}|$ is just $\mathcal{F}$.*

Proof. In Definition A.5.17, we have $|D_{\mathcal{F}}| = \triangle \otimes_{\Delta^{op}} D_{\mathcal{F}} = \int^{[n]\in\Delta^{op}} \Delta^n \otimes D_{\mathcal{F}}([n])$.

For a fixed object c of $\mathcal{C}$, we obtain $|D_{\mathcal{F}}|_c = \int^{[n]\in\Delta^{op}} \Delta^n \otimes D_{\mathcal{F}}([n], c)$. Since $D_{\mathcal{F}}([n], c)$ is just the constant simplicial set of $\mathcal{F}_c([n])$, from the remark above we see it will be isomorphic to $\mathcal{F}_c$. Therefore $|D_{\mathcal{F}}| \cong \mathcal{F}$. □

Lemma 2.1.16. *Under assumptions above, in $sPr(\mathcal{C})_{inj}$ the Bousfield-Kan map $\mathrm{hocolim} D_{\mathcal{F}} \to |D_{\mathcal{F}}|$ is a weak equivalence. And therefore $\mathrm{hocolim} D_{\mathcal{F}}$ is weakly equivalent to $\mathcal{F}$.*

Proof. In $sPr(\mathcal{C})_{inj}$ cofibrations are just objectwise cofibrations and in s**Set** cofibrations are injective maps. Therefore any object $\mathcal{F}$ in $sPr(\mathcal{C})_{inj}$ is cofibrant. Then from Definition A.5.22, for any simplicial object X in $sPr(\mathcal{C})_{inj}$ its homotopy colimit is computed by the coend $N(- \downarrow \Delta^{op})^{op} \otimes_{\Delta^{op}} X$. Fixing the object c of $\mathcal{C}$, X_c will be a simplicial object in s**Set** and its homotopy colimit is just the value of $\mathrm{hocolim} X$ on c. From Corollary A.5.30 we see the map $\mathrm{hocolim} X_c \to |X_c|$ is a weak equivalence. But $|X_c| = |X|(c)$, this means the Bousfield-Kan map $\mathrm{hocolim} X \to |X|$ is an objectwise weak equivalence. Especially when $X = D_{\mathcal{F}}$, $\mathrm{hocolim} D_{\mathcal{F}} \to |D_{\mathcal{F}}|$ is a weak equivalence. □

[1] [41, Exerciese 11 of Chapter I in p63]

The above two lemmas are in [16, Remark 2.1].

Theorem 2.1.17. [2]

(1). In $sPr(\mathcal{C})_{inj}$, a simplicial presheaf $\mathcal{F}$ satisfies descents for any hypercover $\mathcal{U} \to c$ if and only if $\mathbb{R}\mathrm{Map}(c, \mathcal{F}) \to \mathbb{R}\mathrm{Map}(\mathcal{U}, \mathcal{F})$ is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$.

(2). In $sPr(\mathcal{C})_{proj}$, a simplicial presheaf $\mathcal{F}$ satisfies descents for any hypercover $\mathcal{U} \to c$ if and only if $\mathbb{R}\mathrm{Map}(c, \mathcal{F}) \to \mathbb{R}\mathrm{Map}(\mathcal{U}, \mathcal{F})$ is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$.

These mean local objects are just those satisfying descents for hypercovers.

Proof. For (1), we just need to prove $\mathrm{Map}(c, \hat{\mathcal{F}}) \to \mathrm{Map}(\mathcal{U}, \hat{\mathcal{F}})$ is a weak equivalence where $\hat{\mathcal{F}}$ is a fibrant replacement of $\mathcal{F}$. Since the property of satisying descents is also for fibrant obejcts, we can just assume F is fibrant in $sPr(\mathcal{C})_{inj}$.

Let $D_{\mathcal{U}}$ be a simplciial object in $sPr(\mathcal{C})_{inj}$ such that for any $[n]$, $D_{\mathcal{U}}([n])$ is the discrete simplicial presehaf $\mathcal{U}_n$. Then by Lemma 2.1.16, $\mathrm{hocolim} D_{\mathcal{U}} \to |D_{\mathcal{U}}| = \mathcal{U}$ is a weak equivalence. Next according to Proposition A.5.16, the induced map $\mathrm{Map}(\mathcal{U}, \mathcal{F}) \to \mathrm{Map}(\mathrm{hocolim} D_{\mathcal{U}}, \mathcal{F})$ is a weak equivalence. Moreover

$$\mathrm{Map}(\mathrm{hocolim} D_{\mathcal{U}}, \mathcal{F}) \simeq \mathrm{holim}\mathrm{Map}(D_{\mathcal{U}}, \mathcal{F})$$

is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$ which is a corollary of Theorem A.3.8. Then it's clear $\mathrm{Map}(c, \mathcal{F}) = \mathcal{F}(c) \to \mathrm{Map}(\mathcal{U}, \mathcal{F})$ is a weak equivalence if and only if $\mathcal{F}(c) \to \mathrm{holim}\mathrm{Map}(D_{\mathcal{U}}, \mathcal{F})$ is a weak equivalence, i.e. satisfying the descent.

For (2), c is cofibrant in $sPr(\mathcal{C})_{proj}$ from Remark 2.1.3 since trivial fibrations in $s\mathbf{Set}$ are surjective. For $D_{\mathcal{U}}$, $D_{\mathcal{U}}([n]) = \mathcal{U}_n$ is a coproduct of representable presheaves hence being cofibrant in $sPr(\mathcal{C})_{proj}$. Then from Definition A.5.22, its homotopy colimit can be computed as $\mathrm{hocolim} D_{\mathcal{U}} = N(- \downarrow \Delta^{op}) \otimes_{\Delta^{op}} D_{\mathcal{U}}$ which is cofibrant by Remark A.5.21. Similar to Lemma 2.1.16, $\mathrm{hocolim} D_{\mathcal{U}} \to |D_{\mathcal{U}}| = \mathcal{U}$ is a weak equivalence. Therefore $\mathrm{hocolim} D_{\mathcal{U}}$ is a cofibrant replacement of $\mathcal{U}$. The remaining proof is the same as in (1). ☐

Remark 2.1.18. According to Theorem A.6.14, fibrant objects in local model categories are just local objects that are fibrant in original model categories. From Remark A.6.16, the fully faithful functor $\mathbb{R}\mathrm{id} : \mathrm{Ho}(sPr(\mathcal{C})_{loc}) \to \mathrm{Ho}(sPr(\mathcal{C})_{glob})$ makes $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ a full subcategory of $\mathrm{Ho}(sPr(\mathcal{C})_{glob})$ whose essential image consists of all local objects i.e. simplicial presheaves satisfying descents for every hypercover, and we call them *stacks* more precisely *higher stacks* or ∞*-stacks*. $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ is the *category of stacks* which is also denoted by $\mathbf{St}(\mathcal{C})$. The category $\mathrm{Ho}(sPr(\mathcal{C})_{glob})$ is the category of *prestacks* and the *stackification functor* is the left derived functor $\mathbf{a} = \mathbb{L}\mathrm{id}$. Moreover the left identity map $\mathrm{id} : sPr(\mathcal{C})_{glob} \to sPr(\mathcal{C})_{loc}$ commutes with homotopy pullbacks [67, Proposition 3.4.10], since local weak equivalences are preserved by pullbacks along objectwise fibrations and from Theorem A.4.10 homotopy pullbacks can be computed in a way replacing one side by an objectwise fibration.

2.1.1 Truncation

Next we discuss relationships between higher stacks and sheaves or classical stacks defined for example in [71], but we suppose classical stacks are fibered in groupoids.

A simplicial set is n-truncated if it has trivial homotopy groups for $k > n$, and we can define *truncations* for any simplicial model category.

Definition 2.1.19. [3] Suppose $\mathcal{M}$ is a simplicial model category. An object x in $\mathcal{M}$ or $\mathrm{Ho}(\mathcal{M})$ is called n*-truncated* if for any object $y \in \mathcal{M}$, $\mathbb{R}\mathrm{Map}(y, x)$ is n-truncated as a simplicial set. An object x in $\mathcal{M}$ is called *truncated* if it's n-truncated for some integer $n < \infty$.

[2] [16, Lemma4.4]
[3] [67, Definition 3.7.1]

Remark 2.1.20. If $\mathcal{M} = s\mathbf{Set}$, then the definition above is equivalent to the usual one. If for a simplicial set x, it's n-truncated in the sense of the definition above, then we can choose $y = \Delta^0$, then $\mathrm{Map}(\Delta^0, Px) = Px$ where Px is a fibrant replacement of x. Since Px has the some homotopy type of x, this means x is n-truncated as a simplicial set.

Conversely if x is n-truncated as a simplical set, then so is Px. For any map $(\Delta^k, \partial\Delta^k) \to \mathrm{Map}(y, Px)$, it's equivalent to a map $(\Delta^k \times y, \partial\Delta^k \times y) \to Px$. Since Px is a Kan complex, $\mathrm{Map}(y, Px)$ is a Kan complex as well. Therefore its simplicial homotopy groups are the correct homotopy types. There is a simplicial homotopy between the map $(\Delta^k, \partial\Delta^K) \to (Px, *)$ and the trivial map. Applying $- \times y$ to this simplicial homotopy, we see $(\Delta^k \times y, \partial\Delta^k \times y) \to Px$ is homotopically equivalent to a map $y \to Px$ which is just a point in $\mathrm{Map}(y, Px)_0$ that means this map is homotopically trivial.

A generalized version of the remark above for truncated simplicial presheaves is the following proposition which is in [67, Lemma 3.7.6].

Proposition 2.1.21. *The following two statements are equivalent for a simplicial presheaf $\mathcal{F}$ in $sPr(\mathcal{C})_{loc}$.*

(1). $\mathcal{F}$ is an n-truncated object in $sPr(\mathcal{C})_{loc}$.

(2). For any object c of $\mathcal{C}$ and any element $x \in \mathcal{F}_0(c)$, the homotopy group $\pi_k(\mathcal{F}(c), x)$ is trivial for all $k > n$.

There is also an n-truncated local model structure on $sPr(\mathcal{C})$ in [67, Section 3.7], which is denoted by $sPr(\mathcal{C})_{loc}^{\leq n}$. Weak equivalences in it are *local n-equivalences* which means they induce isomorphisms on homotopy sheaves for $i \leq n$. Then $sPr(\mathcal{C})_{loc}^{\leq n}$ will be the left Bousfield localization of $sPr(\mathcal{C})_{loc}$ with respect to $\partial\Delta^i \otimes c \to \Delta^i \otimes c$ for all $i > n$ and objects c of $\mathcal{C}$ [67, Corollary 3.7.4], whose fibrant objects are n-truncated objects which are fibrant in the original local model category.

With the notion of trunction, in the following we will see sheaves are 0-truncated simplicial presehaves and classical stacks are 1-truncated simplicial presheaves. We first talk about the case of sheaves.

In the category $Shv(\mathcal{C})$ of sheaves on a site $\mathcal{C}$, isomorphisms are those injective and locally surjective maps[4]. The set of local isomorphisms (injective and locally surjective) in $Pr(\mathcal{C})$ is denoted by W. Then sheaves are W-local objects since the sheafification functor $\mathbf{a}$ is left adjoint and sends local isomorphisms to isomorphisms in $Shv(\mathcal{C})$. And moreover for any presheaf F, the natural map $F \to \mathbf{a}F$ is actually a local isomorphism. Hence $(Pr(\mathcal{C}), W)$ has good localizations (Definition A.6.2). From Theorem A.6.7 we see $Shv(\mathcal{C})$ is equivalent to $Pr(\mathcal{C})[W^{-1}]$.

Viewing a presheaf as a discrete simplicial presehaf, we obtain an embedding functor $j : Pr(\mathcal{C}) \to sPr(\mathcal{C})$. Discrete simplicial sets have trivial homotopy groups for $k \geq 1$ and hence its image consists of 0-truncated simplicial presheaves. From the definition of local weak equivalences, it's clear j sends local isomorphisms between presheaves to local weak equivalences. Then it induces a functor $Pr(\mathcal{C})[W^{-1}] \to \mathrm{Ho}(sPr(\mathcal{C})_{loc})$. It's actually $Shv(\mathcal{C}) \hookrightarrow Pr(\mathcal{C}) \xrightarrow{\gamma \circ j} \mathrm{Ho}(sPr(\mathcal{C})_{loc})$. [61, Proposition 3.0.3] and [65, Proposition 3.2.7] say this functor is fully faithful. In fact the functor $\pi_0 : sPr(\mathcal{C}) \to Pr(\mathcal{C})$ sending a simplicial presheaf to its homotopy presheaf of path components sends local weak equivalences to local isomorphisms since for a map of presheaves it's a local isomorphism if and only if its sheafification is an isomorphism between sheaves. Then it gives a functor $\mathrm{Ho}(sPr(\mathcal{C})_{loc}) \to Shv(\mathcal{C})$. [67, Proposition 3.7.8] shows this functor is an equivalence between 0-truncated higher stacks and sheaves, whose inverse is the functor we have defined above.

Actually it's very natural to see that sheaves are simplicial presheaves satisfying Čech descent with respect to Čech nerves. But [16, Corollary A.9] shows that for n-truncated simplicial presheaves, they satisfy descent for all hypercovers if and only if they satisfy Čech descent. In this way it's also clear to see sheaves are 0-truncated higher stacks.

As for stacks it's similar to the case of sheaves. [65, Corollary 3.2.5] asserts there is also a fully faithful functor from the category $\mathbf{St}(\mathcal{C})[(1\text{-iso})^{-1}]$ of stacks to $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ whose essential image consists

[4]See [41, Corollary 5 in section 7 chapter III]

of 1-truncated simplicial presheaves. Also in [29, Theorem 3.9], Hollander proves a category fibered in groupoids is a classical stack if and only if it satisfies descent for Čech nerves (this will be equivalent to satisfying descent for all hypercovers [16, Corollary A.9]) though in that article the descent is computed in groupoids. More details of the equivalence between classical stacks and 1-truncated simplicial presheaves can be found in [61, Section 4] and [65, Section 3.2.1].

2.2 Higher Stacks on the Model Site

In the following we do not suppose $\mathcal{C}$ is a site but just a model category in which the set of weak equivalences is denoted by W. Obviously the structure of $sPr(\mathcal{C})_{glob}$ is irrelevant with the model structure on $\mathcal{C}$ which means it does not contain any homotopical information in $\mathcal{C}$. Therefore we want to deal with a model category $sPr(\mathcal{C})$ different from $sPr(\mathcal{C})_{glob}$. For any object c of C, it corresponds to a constant simplicial presheaf which is also denoted by c such that for any integer n, c_n is the representable presheaf of c. Let

$$h_W := \{a \to b \in sPr(\mathcal{C}) | a \to b \in W \subseteq \mathrm{Mor}(\mathcal{C})\}$$

and we define $\mathcal{C}^\wedge := L_{h_W} sPr(\mathcal{C})_{glob}$ the left Bousfield localization of $sPr(C)_{glob}$ with respect to the set h_W. To avoid dealing with a similar condition twice, in the following we just consider the case of $sPr(\mathcal{C})_{proj}$.

Note that viewing an object of $\mathcal{C}$ as a constant simplicial presheaf is a fully faithful embedding $\mathcal{C} \to sPr(\mathcal{C})$. And since the left Bousfield localization will not change the underlying categorical structure, we obtain a fully faithful embedding $h : \mathcal{C} \to \mathcal{C}^\wedge$. Moreover this functor sends weak equivalences to weak equivalences. Then passing to homotopy categories, we have $\mathrm{Ho}(h) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^\wedge)$. Now a natural question arises. Is this functor still fully faithful? The answer is positive and we will solve this problem following [67].

Suppose $\Gamma : \mathcal{C} \to \mathcal{C}^\Delta$ is a *cosimplicial resolution functor* sending an object c of $\mathcal{C}$ to its cosimplicial resolution (see Definition A.4.19) i.e. a Reedy cofibrant replacement of the constant cosimplicial object c. Moreover we assume there is a natural transformation from Γ to the constant cosimplicial functor. In most cases this assumption is satisfied since most model categories we deal with are functorial. Define

$$\underline{h} : \mathcal{C} \to sPr(\mathcal{C}),\ c \mapsto \mathrm{Hom}_{\mathcal{C}}(\Gamma(-), c)$$

If c is fibrant in $\mathcal{C}$, then [28, Proposirion 16.4.6] shows that the simplicial presheaf $\mathrm{Hom}_{\mathcal{C}}(\Gamma(-), c)$ is fibrant in $sPr(\mathcal{C})_{proj}$ and the functor $\underline{h}$ sends any trivial fibration $c \to d$ to a trivial fibration $\mathrm{Hom}_{\mathcal{C}}(\Gamma(-), c) \to \mathrm{Hom}_{\mathcal{C}}(\Gamma(-), d)$. Then a dual version of Proposition A.3.3 implies there is a right derived functor

$$\mathbb{R}\underline{h} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(sPr(\mathcal{C})),\ c \mapsto \underline{h}(Rc)$$

where Rc is a fibrant replacement of c in $\mathcal{C}$. Moreover since trivial fibrations will not be changed in the process of left Bousfield localization, $\underline{h} : \mathcal{C} \to \mathcal{C}^\wedge$ will preserve trivial fibrations as well and there is also a right derived functor. We can also prove $\underline{h} : \mathcal{C} \to \mathcal{C}^\wedge$ preserves fibrant objects.

Theorem A.6.14 tells us that fibrant objects in $\mathcal{C}^\wedge$ are just those fibrant objects in $sPr(\mathcal{C})_{proj}$ which are h_W-local. Suppose $a \to b$ belongs to h_W, it's necessary to prove

$$\mathbb{R}\mathrm{Map}(b, \mathrm{Hom}_{\mathcal{C}}(\Gamma(-), c)) \to \mathbb{R}\mathrm{Map}(a, \mathrm{Hom}_{\mathcal{C}}(\Gamma(-), c))$$

is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$. But since a is a constant simplicial presheaf, a will be cofibrant. So if we assume c is fibrant, the map above will be the following map

$$\mathrm{Hom}_{\mathcal{C}}(\Gamma(b), c) \to \mathrm{Hom}_{\mathcal{C}}(\Gamma(a), c)$$

of simplicial sets. Since in the Reedy model category $\mathcal{C}^\Delta$ weak equivalences are objectwise, $a \to b$ will be a weak equivalence of constant cosimplicial objects. Then it induces a weak equivalence $\Gamma(a) \to \Gamma(b)$ of Reedy cofibrant objects. Then from Lemma A.3.2 and [28, Proposition 16.4.6], the map $\mathrm{Hom}_{\mathcal{C}}(\Gamma(b), c) \to \mathrm{Hom}_{\mathcal{C}}(\Gamma(a), c)$ will be a weak equivalence of simplicial sets.

The statement above implies the following fact.

Lemma 2.2.1. *A simplicial presheaf $\mathcal{F}$ is a h_W-local object if and only if viewing a functor $\mathcal{C}^{op} \to s\mathbf{Set}$, $\mathcal{F}$ preserves weak equivalences.*

Then with Theorem A.6.14 we have

Corollary 2.2.2. *An object $\mathcal{F}$ in $\mathcal{C}^\wedge$ is fibrant if and only if it's objectwise fibrant and preserves weak equivalences when viewing it as $\mathcal{C}^{op} \to s\mathbf{Set}$.*

Theorem 2.2.3 (Model Yoneda's Lemma)**.** [5]

(1). The two functors $\mathrm{Ho}(h)$, $\mathbb{R}\underline{h} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^\wedge)$ *are canonically isomorphic.*

(2). $\mathrm{Ho}(h)$ *and* $\mathbb{R}\underline{h}$ *are fully faithful.*

Proof. For (1), it's [67, Lemma 4.2.2].

For (2), according to the technique of simplicial localization via *left homotopy function complexes*, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(a,b) = \pi_0\mathrm{Map}(a,b) = \pi_0\mathrm{Hom}_{\mathcal{C}}(\Gamma(a), Rb)$$

where $\Gamma(a)$ is a cosimplicial resolution and Rb is a fibrant replacement. But

$$\mathrm{Hom}_{\mathcal{C}}(\Gamma(a), Rb) = \underline{h}(Rb)(a) \cong \mathrm{Map}(a, \underline{h}(Rb))$$

Since a is cofibrant in $\mathcal{C}^\wedge$ and $\underline{h}$ preserves fibrant objects,

$$\pi_0\mathrm{Map}(a, \underline{h}(Rb)) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}^\wedge)}(a, \underline{h}(Rb))$$

Note that $\mathcal{C}^\wedge$ is a simplicial model category and then the equality above can be seen in the proof of Lemma A.5.15. Finally the isomorphism between $\mathrm{Ho}(h)$ and $\mathbb{R}\underline{h}$ proves $\mathbb{R}\underline{h}$ is fully faithful. ☐

Model Yoneda's lemma gets its name for the following corollary.

Corollary 2.2.4. *For any fibrant object $\mathcal{F}$ in $\mathcal{C}^\wedge$ we have*

$$\mathbb{R}\mathrm{Map}(\mathbb{R}\underline{h}(c), \mathcal{F}) \cong \mathbb{R}(\mathrm{Ho}(h)(c), \mathcal{F}) \cong \mathbb{R}\mathrm{Map}(c, \mathcal{F}) \cong \mathcal{F}(c)$$

To apply the theory of simplicial presheaves on a site in last section to the case of model categories, we need to define the concept of topology on a model category.

Definition 2.2.5. A *model pre-topology* τ on a model category $\mathcal{C}$ associates every object c of $\mathcal{C}$ with a set $\mathrm{Cov}_\tau(c)$ of families of maps $\{u_i \to c | i \in I\}$ in $\mathrm{Ho}(\mathcal{C})$ such that

(1). (Stability) For any object c of $\mathcal{C}$ and any isomorphism $a \to c$ in $\mathrm{Ho}(\mathcal{C})$, $\{a \to c\}$ is in $\mathrm{Cov}_\tau(c)$.

(2). (Composition) If $\{u_i \to c | i \in I\}$ belongs to $\mathrm{Cov}_\tau(c)$ and for any i, $\{v_{ij} \to u_i | j \in J_i\}$ is in $\mathrm{Cov}_\tau(u_i)$, then the family $\{v_{ij} \to c | i \in I,\ j \in J_i\}$ is an element of $\mathrm{Cov}_\tau(c)$.

(3). (Homotopy base change) Assume the two conditions above hold. For any $\{u_i \to c | i \in I\}$ in $\mathrm{Cov}_\tau(c)$ and any morphism $a \to c$ in $\mathrm{Ho}(\mathcal{C})$, the family $\{u_i \times^h_c a \to a | i \in I\}$ is in $\mathrm{Cov}_\tau(a)$ where $- \times^h -$ means homotopy pullbacks.

A model category with a model pre-topology $(\mathcal{C}, \tau)$ is called a *model site*.

Remark 2.2.6. $u_i \times^h_c a$ is the homotopy pullback of $u_i \to c \leftarrow a$ in $\mathrm{Ho}(\mathcal{C})$. Note that in $\mathrm{Ho}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(a,c) = [Qa, Rc] = \mathrm{Hom}_{\mathcal{C}}(Qa, Rc)/\sim$$

where Qa is a cofibrant replacement and Rc is a fibrant replacement. To compute the homotopy pulllback above, we can find a lifting $Qu_i \to Rc \leftarrow Qa$ and compute this homotopy pullback which is only up to non-canonical isomorphisms in $\mathrm{Ho}(\mathcal{C})$.

[5] [67, Lemma 4.2.2 and Theorem 4.2.3]

In the classical case, there is an equivalence between Grothendieck pre-topology and Grothendieck topology. Details can be found in [41, Section III.2]. We also have a similar relation here. Any model pre-topology τ can induces a Grothendieck topology on $\mathrm{Ho}(\mathcal{C})$ natually such that a sieve $\{u_i \to c | i \in I\}$ is a covering sieve if and only if it contains some element in $\mathrm{Cov}_\tau(c)$. Conversely any Grothendieck topology can also induce a model pre-topology. In this sense, the two notions are equivalent.

Higher prestacks on a model site $(\mathcal{C}, \tau)$ are just objects in the category of $\mathrm{Ho}(\mathcal{C}^\wedge)$ or simply objects in $\mathcal{C}^\wedge$. To obtain higher stacks on the model site, we also need concepts of hypercovers and descent. And then we can do left Bousfield localization with respect to these hypercovers. In this way, weak equivalences we obtain will be *homotopy local weak equivalences* and we also clled them *π_*-equivalences*.

Remark 2.2.7 (π_*-equivalences)**.** Suppose $(\mathcal{C}, \tau)$ is a model site. $\mathcal{F}$ is a prestack in $C^\wedge$ and $R\mathcal{F}$ is a fibrant replacement in $\mathcal{C}^\wedge$. $\pi_0^{pr}(R\mathcal{F})$ is a presheaf on $\mathcal{C}$ sending any object c of $\mathcal{C}$ to the set $\pi_0(R\mathcal{F}(c))$ of components. Since fibrant objects in $\mathcal{C}^\wedge$ are fibrant in $sPr(\mathcal{C})$, there will exists objectwise weak equivalences between any two different fibrant replacements. Then $\pi_0^{pr}(R\mathcal{F})$ is well defined up to isomorphism and we get a functor

$$\pi_0^{eq} : \mathcal{C}^\wedge \to Pr(\mathcal{C}), \;\; \mathcal{F} \mapsto \pi_0^{pr}(R\mathcal{F})$$

A weak equivalence between $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}\wedge$ will induce a weak equivalence between $R\mathcal{F}$ and $R\mathcal{G}$. But they are h_W-local. From Lemma A.6.13, it will be a weak equivalence in $sPr(\mathcal{C})$. And then the induced morphism between $\pi_0^{pr}(\mathcal{F})$ and $\pi_0^{pr}(\mathcal{G})$ will be an isomorphism. Therefore π_0^{eq} factors through $\mathrm{Ho}(\mathcal{C})$.

On the other hand for any weak equivalence $a \to b$ in $\mathcal{C}$, it will be a weak equivalence in $\mathcal{C}^\wedge$. Then from Proposition A.5.16, $\mathrm{Map}(b, R\mathcal{F}) \to \mathrm{Map}(a, R\mathcal{F})$ will be a weak equivalence of simplicial sets. Especially at the level of 0, it's an isomorphism. This means $\pi_0^{pr}(R\mathcal{F})$ factors through $\mathrm{Ho}(\mathcal{C})^{op}$. Therefore we can view π_0^{eq} as the functor

$$\pi_0^{eq} : \mathrm{Ho}(\mathcal{C}^\wedge) \to Pr(\mathrm{Ho}(\mathcal{C}))$$

.

As we have stated, any model pre-topology can induce a Grothendieck topology in $\mathrm{Ho}(\mathcal{C})$. We can the do sheafification on $\mathrm{Ho}(\mathcal{C})$ and we obtain

$$\pi_0^\tau : \mathrm{Ho}(\mathcal{C}^\wedge) \to Shv(\mathrm{Ho}(\mathcal{C}))$$

If τ is clear, we can omit this symbol.

For higher cases it's more subtle. Just like the definition of local weak equivalences, we work on $\mathcal{C}/c$ for an object c of $\mathcal{C}$ but here we need to suppose c is fibrant in $\mathcal{C}$. If $\mathcal{C}$ is right proper, then this assumption is not necessary. We explain the reason. In $\mathcal{C}/c$ a morphism is a weak equivalence, a cofibration or a fibration if its image in $\mathcal{C}$ is such one. Then a fibrant object in $\mathcal{C}/c$ is just a fibration in $\mathcal{C}$ with codomain c. Then for an object $a \to c$ in $\mathcal{C}/c$, we replace it by its fibrant replacement via the factorization $a \to b \to c$ in C such that $b \to c$ is a fibration. If we suppose c is fibrant, then b will be fibrant. From a dual version of Proposition A.4.7 and Remark A.4.8, we conclude in this case the homotopy pullback in $\mathcal{C}/c$ will be preserved when passing to $\mathcal{C}$. But according to Theorem A.4.10, if $\mathcal{C}$ is right proper, homotopy pullbacks will be preserved no matter c is fibrant or not. In this way we can define a pre-topology in $\mathrm{Ho}(\mathcal{C}/c)$ such that a family of maps is a covering if its image in $\mathrm{Ho}(\mathcal{C})$ is a covering. This will satisfy axioms of model pre-topology. The remainder will be similar to the previous one. For any prestack $\mathcal{F}$, any object c in $\mathcal{C}$ and any element $x \in \mathcal{F}_0(c)$, we have a homotopy sheaf $\pi_n(\mathcal{F}|c, x)$ on $\mathrm{Ho}(\mathcal{C}/c)$.

π_*-equivalences are just those maps inducing isomorphisms on these homotopy sheaves.

The definition of $\pi_n(\mathcal{F}|c, x)$ in [67, Definition 4.5.3] looks different from what we state here but they are equivalent. Actually there is an effective way to compute higher homotopy groups for a Kan complex, in which we can only compute the set of components. For any Kan complex X and $x \in X_0$,

$$\pi_n(X, x) = \pi_0 F_n(X)_x$$

where $F_n(X)_x = \Delta^0 \times_{X^{\partial\Delta^n}} X^{\partial\Delta^n}$. Details can be found in [35, Section 4.1].

To obtain the model category of higher stacks on the model site with weak equivalences being π_*-equivalences, we need to do left Bousdield localization on $\mathcal{C}^\wedge$ with respect to a certain small set of "hypercovers". The notion of hypercovers here is really subtle. From Proposition 2.1 we know in $sPr(\mathcal{C})$, a morphism $\mathcal{F} \to \mathcal{G}$ is a local trivial fibration if $\mathcal{F}^{\Delta^n} \to \mathcal{F}^{\partial\Delta^n} \times_{\mathcal{G}^{\partial\Delta^n}} \mathcal{G}^{\Delta^n}$ is a local epimorphism at the level of 0. But for a model site $\mathcal{C}$ the Grothendieck topology is on $\mathrm{Ho}(\mathcal{C})$. So we need to say a local epimorphism at the level of homotopy. In Definition 2.1.4 of local trivial fibrations, commutative diagrams are strict. But here commutative diagrams implying lifting properties should be up to homotopy.

Definition 2.2.8. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{Ho}(\mathcal{C}^\wedge)$ is called a *τ-covering* if the induced morphism $\pi_0(\mathcal{F}) \to \pi_0(\mathcal{G})$ is an epimorphism of sheaves.

[67, Proposition 3.1.4] characterizes τ-coverings as *homotopy locally surjective* morphisms. Therefore using this concept we can obtain a homotopy analogue of local trivial fibrations. So a *homotopy τ-covering* for a fibrant object c of $\mathcal{C}$ can be defined to be a morphism $\mathcal{U} \to c$ of simplicial presheaves such that

(1). for each $n, \mathcal{U}_n$ is a coproduct of representable presheaves

(2). $\mathcal{U}^{\Delta^n} \to \mathcal{U}^{\partial\Delta^n} \times^h_{c^{\partial\Delta^n}} c^{\Delta^n}$ is a covering in $\mathrm{Ho}(\mathcal{C}^\wedge)$.

Note that there is a more general definition of homotopy τ-coverings concerning simplicial objects in $sPr(\mathcal{C})$. It's in [67, Definition 3.2.3]. Then we can do left Bousfield localization with respect to a certain small set of τ-hypercovers. [67, Theorem 4.6.1] shows elements in this small set can be *pseudo-representable* hypercovers ([67, Definition 4.4.1]). Then in this local model category we obtain, weak equivalences are just π_*-equivalences and fibrant objects are those fibrant in $\mathcal{C}^\wedge$ while satisfying *hyperdescent* for hypercovers [67, Definition 4.6.5 and Corollary 4.6.3].

Here the local model category is denoted by $\mathcal{C}^{\sim,\tau}$. Objects in $\mathrm{Ho}(\mathcal{C}^{\sim,\tau})$, which is also denoted by $\mathbf{St}(\mathcal{C},\tau)$ and if τ is clear, it can be omitted, are called higher stacks on the model site.

2.3 Geometric Stacks

In this section we study how we can apply the theory of simplicial presheaves to algebraic geometry. As we have said sheaves are 0-truncated higher stacks and usual stacks are 1-truncated higher stacks. In algebraic geometry a special class of stacks are concerned i.e. *Deligne-Mumford stacks* or *Artin stacks*. The other name for Artin stacks is *algebraic stacks*. These stacks play an important role in the moduli theory and they have some geometric nature. To begin with this section, we first talk about definitions for two stacks above[6] and then we consider their generalization for higher cases. In the next section we study its derived case i.e. derived algebraic geometry.

2.3.1 Classical Geometric Stacks

In the category $\mathbf{Sch}_k$ of schemes where k is a commutative ring there are some useful Grothendieck topologies which are Zariski topology (zar), étale topology (et), locally of finite presentation and faithfully flat topology (fppf) and faithfully flat and quasi-compact topology (fpqc).[7] For definitions you can look at [71, Section 2.3]. Relations of these topologies are such that

$$(\text{zar}) \preceq (\text{et}) \preceq (\text{fppf}) \preceq (\text{fpqc})$$

Actually we can only work on $\mathbf{Aff}_k$ the category of affine schemes since every scheme can be covered by affine schemes and the theory here is local[8]. If we do not remind specially $\mathbf{Sch}_k$ and $\mathbf{Aff}_k$ in this section are always equipped with **étale topology**, though Zariski topology is enough for our purpose here.

[6]Limited by the space, we have to leave out many details. For basic algebraic geometry readers can consult with [8] or [3] if necessary. For details of DM-stacks or algebraic stacks [5], [46, Section 2.2] and [47] contain much important information.

[7]fppf gets its name for "fidèlement plat et de présentation finie" and fpqc for "fidèlement plat et quasi-compact".

[8]For example [71, Theorem 2.44] shows any scheme when viewed as a representable functor is a sheaf on the fpqc site $\mathbf{Sch}_k^{\text{fpqc}}$. Especially it will be a sheaf on the étale site.

Remark 2.3.1 (Philosophy of étale topology)**.** This remark is to make the abstract concept étale topology more accessible. But limited by my level of knowledge, it's based on my own understanding and with the help of Chatgpt.

The word "étale" in French is often used to describe the flatness or peace of the sea. It appears many times in French literature. We can give some examples in the following

- Le vent avait cessé, la mer était **étale**, comme si elle eût été recouverte d'une immense nappe de cristal bleu, dont on n'eût laissé voir, çà et là, que quelques plis maladroits et presque invisibles. (Marcel Proust, *À la recherche du temps perdu*)
- La mer était d'huile, **étale** comme une table; Le soleil sur l'eau faisait une tache d'or; Aucun souffle n'agitait sa vaste étendue, Et la barque immobile, au loin, semblait perdue. (Victor-Marie Hugo, *Les Contemplations*)
- La mer était **étale**, calme. Des tas de bois flottaient à côté de nous. La plage était déserte et le soleil commençait à descendre. (Albert Camus, *L'Étranger*)

I think the following painting of a Russian marine painter Ivan Aivazovsky shows the meaning of the word "étale" vividly. If you are interested in his other works, you can look at his wiki page.

Figure 1: The Black Sea at night

Since "étale" is often used to describe the sea not a river. Rivers always have branches but the sea doesn't have that. Therefore "étale" naturally means flat and unramified, in the language of geometry which means an étale map will not change the shape of a geometric object locally in the sense that it induces an isomorphism between tangent spaces. I hope discussions above can make étale topology more vividly.

In algebraic geometry étale topology is famous for its cohomology theory which can be used to solve Weil's conjecture.[9] Its origin can be traced bake to the notion of *isotrivial covers* due to Serre from which Serre successfully defines the first Weil cohomology group. As Weil has observed, for polynomial equations with integer coefficients the complex topology of the set of complex solutions of these equations profoundly

[9] A strict theory can be found in [45].

influence the number of solutions of these equations modulo a prime number. So here étale topology is to serve just like the complex topology but it's not dependent on the complex structure of a geometric object. We give an example here to show advantages of complex topology and the necessity of étale maps which is in [75, Section 2.2].

Consider a function $\mathbb{C}^\times \to \mathbb{C}^\times$ sending z to z^2. For usual complex topology this function has the inverse locally but not globally. Therefore the map

$$\mathcal{O}_{\mathbb{C}}^\times \to \mathcal{O}_{\mathbb{C}}^\times, \ \ f \mapsto f^2$$

of sheaves is surjective where $\mathcal{O}_{\mathbb{C}}^\times$ is the sheaf of units of the sheaf $\mathcal{O}_{\mathbb{C}}$ of holomorphic functions. Similarly the map

$$\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^\times, \ \ f \mapsto \exp(f)$$

of sheaves is surjective, whose kernel is the constant sheaf $\underline{2\pi i\mathbb{Z}}$ on $\mathbb{C}$. For nice enough spaces the sheaf cohomology of local systems is naturally isomorphic to the singular cohomology with coefficients of this space [72, Theorem 11.13]. Therefore from the short exact sequence

$$0 \longrightarrow \underline{2\pi i\mathbb{Z}} \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}}^\times \longrightarrow 0$$

the induced long exact sequence will show connections between singular cohomology and coherent cohomology.

However it's not in the case of Zariski topology. In the Zariski topology the map $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}^\times \to \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}^\times$ of sheaves sending f to f^2 is not surjective where $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\mathbb{C}[X]$, since the element $X \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}^\times(D(X))$ does not admit square roots locally in the Zariski topology. But if we consider the integral extension

$$\mathbb{C}[X, X^{-1}] \to \mathbb{C}[X, X^{-1}, Y]/(Y - X^2) = A$$

which induces a surjective map $\mathrm{Spec}A \to D(X)$ on topological spaces, $X|\mathrm{Spec}A$ will admit a square root Y. Moreover if we consider the complex topology on $\mathbb{C}$-points, $\mathbb{C}$-points on $\mathrm{Spec}A$ are maximal ideals of the form $\mathfrak{m} = (Y - z, Y^2 - X)$ and their images in $\mathbb{C}$-points of $D(X)$ are just $\mathfrak{n} = (X - z^2)$. Therefore it induces the map sending z to z^2 which is a local isomorphism. Hence to obtain the result we want, we can view such map $\mathrm{Spec}A \to D(X)$ as a generalized open cover. Actually for algebraic varieties over $\mathbb{C}$, a map between them is étale if and only if the induced map on $\mathbb{C}$-points with the complex topology is locally an open embedding, i.e. an étale map between topological spaces.

Now let us introduce the strict definition of étale morphisms.

Definition 2.3.2. (1). A *closed immersion* $i : S_0 \to S$ of schemes is a *first order thickening* of S_0, if the corresponding ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_S \to i^*\mathcal{O}_{S_0})$ satisfies $\mathcal{I}^2 = 0$

(2). A morphism $f : X \to S$ of schemes is called *formally smooth* (resp. *formally étale*, resp. *formally unramified*) if for all following commutative diagrams,

$$\begin{array}{ccc} T_0 & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{u}{\nearrow} & \big\downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

where $i : T_0 \hookrightarrow T$ is a first order thickening of affine schemes, there is a lift $u : T \to X$ making the diagram commutative (resp. there is exactly one such lift u, resp. there is at most one such u).

(3). A morphism $f : X \to S$ of schemes is called *smooth* (resp. *étale*, resp. *unramified*) if it's *locally of finite presentation* and formally smooth (resp. formally étale, resp. formally unramified).

There are many equivalent definitions of these maps in any textbook about algebraic geometry such as in [25] or [3]. But from this lifting definition, it's clear to see étale maps are stable under compositions and pushouts. From this definition it's so clear to see **étale=smooth+unramified**. But there is another characterization **étale=flat+unramified** [8, Corollary 8.5/18] which is not clear.

Remark 2.3.3. The definition of étale morphisms used by Bertrand Toën is a bit different from we state here but they are equivalent. For Toën a map $A \to B$ of commutative algebras is étale if it's locally of finite presentation, B is flat over A via this map and B is flat as a $B \otimes_A B$-module. Actually from [8, Theorem 8.4/3], a morphism $X \to S$ of schemes which is locally of finite presentation is unramified if and only if the diagonal morphism $\Delta : X \to X \times_S X$ is open. But moreover in algebraic geometry we have a theorem which asserts a flat morphism that is locally of finite presentation is open [3, Tag01UA]. Then the equivalence will be clear.

Definition 2.3.4. An *algebraic space* is a sheaf $\mathcal{X}$ on the site $\mathbf{Sch}_k$ such that

(1). for all schemes X, Y and any morphisms $X \to \mathcal{X}, Y \to \mathcal{X}$ of sheaves, the sheaf $X \times_{\mathcal{X}} Y$ is representable by a scheme;

(2). there exists a scheme X called an *atlas* and a surjective étale morphism $X \to \mathcal{X}$, which means for any other map $Y \to \mathcal{X}$ of sheaves where Y is a scheme the projection $X \times_{\mathcal{X}} Y \to Y$ is a suejective étale morphism of schemes.

Actually from an atlas $X \to \mathcal{X}$ we can characterize an algebraic space as a quotient of a scheme by an *étale equivalence* [47, Section 5.2].

Remark 2.3.5. There is a trivial way to view a sheaf as a stack. We say a stack $\mathcal{F}$ is *representable* by an algebraic space $\mathcal{X}$ (resp. scheme X) if there exists an algebraic space $\mathcal{X}$ (resp. scheme X) such that $\mathcal{F}$ is isomorphic to the stack associated with $\mathcal{X}$ (resp. X).

A morphism $\mathcal{F} \to \mathcal{G}$ of stacks is *representable* by algebraic spaces (resp. schemes) if for any object X of $\mathbf{Sch}_k$ and any morphism $X \to \mathcal{G}$ of stacks, the pullback $X \times_{\mathcal{G}} \mathcal{F}$ is representable by an algebraic space (resp. scheme).

We can do geometry on representable morphisms of stacks. We say a representable morphism $\mathcal{F} \to \mathcal{G}$ has some geometric property such as surjective, flat, smooth, quasi-compact and so on if for any object X of $\mathbf{Sch}_k$, $\mathcal{X} \times_{\mathcal{G}} \mathcal{F} \to X$ has this property.

Definition 2.3.6 (Artin stack)**.** A stack $\mathcal{F}$ on the site $\mathbf{Sch}_k$ is called an *Artin stack* or *algebraic stack* if

(1). the diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by algebraic spaces and quasi-compact;

(2). there exists a scheme X called an *atlas* and a surjective smooth morphism $X \to \mathcal{X}$

[46, Proposition 2.44] shows there are other characterizations for the diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ to be representable. It's equivalent to that for any algebraic space $\mathcal{X}$ morphisms $\mathcal{X} \to \mathcal{F}$ are representable by algebraic spaces or for any scheme X morphisms $X \to \mathcal{F}$ are representable by algebraic spaces. Therefore the second part in the definition makes sense.

Definition 2.3.7 (Deligne-Mumford stack)**.** A stack $\mathcal{F}$ on the $\mathbf{Sch}_k$ is called a *Deligne-Mumford stack* or *DM-stack* simply if

(1). the diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by schemes, quasi-compact and separated;

(2). there exists a scheme X called an *atlas* and a surjective étale morphism $X \to \mathcal{X}$

Example 2.3.8. There are some famous examples of moduli stacks in algebraic geometry being DM-stacks or algebraic stacks. The moduli stack $\mathscr{M}_{g,n}$ classifying smooth, connected and projective n-pointed curves of genus g over the complex field $\mathbb{C}$ is a DM-stack and its compactification $\overline{\mathscr{M}}_{g,n}$ via stable reduction classifying n-pointed stable curves is a proper smooth DM-stack of dimension $3g - 3 + n$ if $2g - 2 + n > 0$. Moreover its coarse moduli space $\overline{M}_{g,n}$ is a projective algebraic space.

On a projective scheme X, moduli stacks $\mathscr{C}oh_X$ classifying coherent $\mathcal{O}_X$-modules and $\mathscr{B}un_X^n$ classifying vector bundles of rank n over X are all algebraic stacks. If X is a smooth projective curve of genus g, then $\mathscr{B}un_X^n$ will be smooth of dimension $n^2(g-1)$. But for higher dimensions such as a smooth projective surface, $\mathscr{B}un_X^n$ may not be smooth any more.

It's not our purpose in this thesis to talk about these moduli stacks in detail which will make us go too far away from the main line. On the other hand to understand these examples clearly one needs more backgrounds on algebraic geometry than we assumed for reading this thesis. But if a reader is interested in them, he or she can be benefited a lot from [5] or [46].

2.3.2 Higher Geometric Stacks

In the following we will consider the generalization of algebraicity for higher stacks so that we can do geometry on them. Our main references here are [68] and [63, Chapter 3]. But in these two references names for this concept are different. In the former it's a special case of *geometric stacks* while in the latter it's called *n-algebraic stacks*. This generalization can be dated back to C. Simpson. For simplicity we work on $\mathbf{Aff}_k$ of affine schemes with étale topology here.

From last two sections we have two ways to obtain the natural embedding $\mathbf{Aff}_k \to \mathbf{St}_k$ where $\mathbf{St}_k = \mathrm{Ho}(sPr(\mathbf{Aff}_k)_{loc})$. The first one is to view an affine scheme as a sheaf [71, Theorem 2.44] and any sheaf is actually a 0-truncated higher stack. The second one is to view $\mathbf{Aff}_k$ as a trivial model category where weak equivalences are isomorphisms and all morphisms are fibrations and cofibrations. Then from Theorem 2.2.3 and étale descent (sheaf condition) the fully faithful functor exists. Objects in $\mathbf{St}_k$ isomorphic to some $\mathrm{Spec}A$ are called *affine schemes* where A is any k-algebra.

Based on different notions of morphisms satisfying certain conditions, we will obtain different n-geometric stacks. For example Artin stacks have an atlas of a smooth morphism. Then to obtain Artin n-stacks the choice of morphisms here should be smooth morphisms. For DM-stacks the choince should be étale morphisms. Moreover for schemes, the choice can be Zariski open immersions. Next we introduce this idea precisely.

Definition 2.3.9. [10] Let $\mathbf{P}$ be a class of morphisms in $\mathbf{Aff}_k^\tau$. We say $\mathbf{P}$ is *τ-local* if it satisfies the following properties.

(1). For any τ-covering $\{U_i \to X | i \in I\}$ in $\mathbf{Aff}_k^\tau$, the morphism $U_i \to X$ is in $\mathbf{P}$.

(2). Morphisms in $\mathbf{P}$ are stable by compositions, equivalences and base changes.

(3). Let $f : X \to Y$ be a morphism in $\mathbf{Aff}_k^\tau$. If there exists a τ-covering $\{U_i \to X | i \in I\}$ such that the composition $U_i \to Y$ lies in $\mathbf{P}$ for all $i \in I$, then f belongs to $\mathbf{P}$.

(4). For any two objects X and Y in $\mathbf{Aff}_k^\tau$, the two morphisms $X \to X \coprod Y$ and $Y \to X \coprod Y$ are in $\mathbf{P}$.

Via the embedding $\mathbf{Aff}_k^\tau \to \mathbf{St}(\mathbf{Aff}_k, \tau)$, we can also view $\mathbf{P}$ as a class of morphisms in $\mathbf{St}(\mathbf{Aff}_k, \tau)$.

Definition 2.3.10 (Geometric stacks)**.** [11] Let $\mathbf{P}$ be a class of morphisms in $\mathbf{Aff}_k^\tau$ which is τ-local.

(1). A stack $\mathcal{F}$ is (-1)-*geometric* if it's an affine scheme i.e. representable stack.

(2). A morphism $\mathcal{F} \to \mathcal{G}$ between stacks is (-1)-*representable* if for any affine scheme X and any morphism $X \to \mathcal{G}$, the homotopy pullback $\mathcal{F} \times_{\mathcal{G}}^h X$ is an affine scheme.

(3). A morphism $\mathcal{F} \to \mathcal{G}$ of stacks is in (-1)-$\mathbf{P}$ if it's (-1)-representable and for any morphism $X \to \mathcal{G}$ where X is an affine scheme, the induced morphism $\mathcal{F} \times_{\mathcal{G}}^h X \to X$ is a $\mathbf{P}$-morphism of affine schemes.

Now let $n \geq 0$.

[10] [68, Assumption 1.3.2.11]
[11] [68, Definition 1.3.3.1]

(1.) Let $\mathcal{F}$ be a stack. An *n-atlas* for $\mathcal{F}$ is a family of morphisms $\{U_i \to \mathcal{F} | i \in I\}$ such that

(a). every U_i is an affine scheme;

(b). every morphism $U_i \to \mathcal{F}$ is in $(n-1)$-$\mathbf{P}$;

(c). the total morphism $\coprod_{i \in I} U_i \to \mathcal{F}$ is an epimorphism.

(2). A stack $\mathcal{F}$ is *n-geometric* if it satisfies the following two conditions.

(a). The diagonal morphism $\mathcal{F} \to \mathcal{F} \times^h \mathcal{F}$ is $(n-1)$-representable.

(b). The stack $\mathcal{F}$ admits an n-atlas.

(3). A morphism $\mathcal{F} \to \mathcal{G}$ of stacks is *n-representable* if for any morphism $X \to \mathcal{G}$ where X is an affine scheme, the homotopy pullback $\mathcal{F} \times^h_{\mathcal{G}} X$ is n-geometric.

(4). A morphism $\mathcal{F} \to \mathcal{G}$ of stacks is in n-$\mathbf{P}$ if it's n-representable and for any morphism $X \to \mathcal{G}$ where X is an affine scheme, there exists an n-atlas $\{U_i\}$ for $\mathcal{F} \times^h_{\mathcal{G}} X$ such that the composition $U_i \to X$ is in $\mathbf{P}$ for every i.

Remark 2.3.11. In [63, Section 3.1] Toën first lets $\mathbf{P}$ be the class of Zariski open immersions and do the gluing process in $\mathbf{St}_k$ to obtain the concept of schemes which can be regarded as a stack admitting a Zariski 0-atlas.[12] Viewing schemes as 0-representable objects, he then continues the gluing process like that for n-geometric stacks in $\mathbf{St}_k$ where the class of morphisms $\mathbf{P}$ changes to be smooth morphisms. In this way he obtains the concept of *n-algebraic stacks*. If we do the gluing process in $\mathbf{St}_k$ directly with $\mathbf{P}$ being the property of smoothness to obtain n-geometric stacks, the concept of 0-geometric stacks will be slightly stronger than the classical algebraic stacks since it requires an affine diagonal. Moreover in this case, arbitrary schemes are actually 1-geometric. There is an example in [50, Remark 1.27]. The scheme X glued by two copies of $\mathbb{A}^2$ with intersection $\mathbb{A}^2 \setminus 0$ does not have an affine diagonal. So it's only 1-geometric.

However, here we seldom distinguish the two definitions. Also we call n-geometric stacks as *n-Artin stacks*. Many examples of n-algebraic stacks or n-geometric stacks can be found in [63, Section 3.2] or [62, Section 3.2]. There is also another way using *hypergroupoids* to characterize geometric stacks due to Pridham in [50] but we will not talk about it here.

Note that [68, Lemma 2.1.1.2] shows an n-geometric stack is $(n+1)$-truncated so that n-Artin stacks and Artin n-stacks are a bit different. For an Artin n-stack we mean it's n-truncated and m-geometric for some m. This notion is more useful and it explains why we do not distinguish the two definitions for n-Artin stacks, since we can suppose n is big enough.

The theory of geometric stacks talked here is covered by the derived case and we are now going to enter the derived world.

2.4 Commutative Differential Graded Algebra

Actually the theory of geometric stacks above can be defined in a more general context. In the last section we suppose the category $\mathbf{Aff}_k$ of affine schemes is equipped with the trivial model structure. Note that any k-algebra is a commutative monoid in the category $\mathbf{Mod}_k$ of k-modules. All commutative monoids in $\mathbf{Mod}_k$ form the category $\mathbf{Alg}_k$ of commutative k-algebras. Here all model structures are trivial. But this motivates us to consider a non-trivial symmetric monoidal model category $\mathcal{C}$ and its commutative monoids. The opposite of the category of commutative monoids in $\mathcal{C}$ is denoted by $\mathbf{Aff}_{\mathcal{C}}$.

The first question we meet is that $\mathbf{Aff}_{\mathcal{C}}$ may not admit a model structure inheriting from $\mathcal{C}$. For example commutative monoids in the category $\mathbf{Ch}(k)$ of chain complexes are commutative differential graded algebras (cdga). But it's well known that if k is a field of characteristic $\mathrm{char}(k) = p > 0$, then the category of cdgas will not admit a model structure inheriting from $\mathbf{Ch}(k)$. Details can be found in [73, Section 5.1]. Therefore in this section we always suppose k **is a field of characteristic** 0 or a $\mathbb{Q}$-algebra if not specified.

[12] For the derived case see Definition 2.5.8.

Also in [73] David White provides some conditions for a monoidal model category $\mathcal{C}$ so that the category $\mathrm{Comm}(\mathcal{C})$ of commutative monoids in it inherits a model structure from $\mathcal{C}$, which means a map in $\mathrm{Comm}(\mathcal{C})$ is a weak equivalence or a fibrations if it's such one in $\mathcal{C}$.

Of course to do homotopical algebraic geometry on $(\mathcal{C}, \mathrm{Comm}(\mathcal{C}))$, $\mathcal{C}$ needs to satisfy some other assumptions and details can be found in [68, Chapter 1.1]. Here we ignore these homotopical difficulties. In [68, Definition 1.3.2.13] there is a definition of *Homotopical Algebraic Geometry context* (simply *HAG context*) so that we can do homotopical algebraic geometry on it. Roughly speaking a HAG context $(\mathcal{C}, \mathbf{Aff}_{\mathcal{C}}, \tau, \mathbf{P})$ consists of a symmetric monoidal model category $\mathcal{C}$, the opposite $\mathbf{Aff}_{\mathcal{C}}$ of the category of commutative monoids in $\mathcal{C}$, a pre-topology τ on $\mathbf{Aff}_{\mathcal{C}}$ and a τ-local class $\mathbf{P}$ of morphisms. Note that here the definition for τ-local class is a bit different from Definition 2.3.9. In the fourth condition of Definition 2.3.9, the coproduct $X \coprod Y$ should be changed to be the homotopy coproduct $X \coprod^{\mathbb{L}} Y$ and moreover $\mathbf{P}$ should be stable under homotopy pullbacks.

2.4.1 Symmetric Monoidal Model Structure for Complexes

In this section we let $\mathcal{C}$ be $\mathbf{Ch}(R)^{\leq 0}$ the category of non-positive cochain complexes. Note that there is a natural way to identify a chain complex and a cochain complex so that $\mathbf{Ch}(k)^{\leq 0}$ is equivalent to $\mathbf{Ch}(R)_{\geq 0}$. For a cochain complex $(C^\bullet, d^n_{C^\bullet})$ we use $C[n]^\bullet$ to denote its shifted complex such that $C[n]^m = C^{n+m}$ with $d^m_{C[n]^\bullet} = (-1)^n \cdot d^{n+m}$. For a complex if whether it's a cochain complex or a chain complex is clear, we will omit the symbol of $\bullet$.

There is a projective model structure on $\mathbf{Ch}(R)^{\leq 0}$ with R being any commutative ring, where a map $f : A^\bullet \to B^\bullet$ is a fibration if f^n is surjective for all $n < 0$, a weak equivalence if f is a quasi-isomorphism i.e. inducing isomorphisms on homology groups and a cofibration if every f^n is injective for $n \leq 0$ and cokernels $\mathrm{coker} f^n$ are projective k-modules. A detailed proof can be found in [33, Lecture 01].

We can also regard a chain complex in $\mathbf{Ch}(R)_{\geq 0}$ as a simplicial R-module thanks to *Dold-Kan correspondence*.

Theorem 2.4.1 (Quillen). [13] *For any category $\mathcal{C}$ of "algebras" (a category of models for Lawvere theory), the category $s\mathcal{C} = \mathcal{C}^{\Delta^{op}}$ of simplicial algebras admits a simplicial model category structure such that*

(1). weak equivalences are those maps which are weak equivalences on the underlying category of simplicial sets

(2). fibrations are those maps that are Kan fibrations on the underlying simplicial sets

(3). cofibrations are retracts of free maps.

Then $s\mathbf{Ab}$, $s\mathbf{Alg}_R$ and $s\mathbf{Mod}_R$ are all simplicial model categories. Dold-Kan correspondence is then a Quillen equivalence

$$N : s\mathbf{Ab} \rightleftarrows \mathbf{Ch}_{\geq 0}(\mathbb{Z}) : \Gamma$$

between simplicial abelian groups and non-negative chain complexes. Note that N and Γ can be right Quillen and left Quillen respectively as well. The equivalence at the level of R-modules can be obtained via the above one. Moreover Dold-Kan correspondence can induce a Quillen equivalence between simplicial algebras and differential graded algebras [58, Theorem 1.1]. But as we have stated when k is a field of characteristic not 0, there will not exist a model structure on $\mathbf{cdgA}_k^{\leq 0}$ inheriting from $\mathbf{Ch}(k)^{\leq 0}$. Hence the equivalence between simplicial commutative rings and cdgas only exists when k is a $\mathbb{Q}$-algebra. In this case we can do homotopical algebraic geometry no matter using simplicial commutative rings or cdgas since they are equivalent.

Now we begin with an introduction to the *closed symmetric monoidal structure* on $\mathbf{Ch}(R)^\bullet$ in detail since the structure on $\mathbf{Ch}(R)^{\leq 0}$ inherits from it.

[13] [52, II.4: Theorem 4, Remarks 1., Remarks 4.]

There is a tensor product $\otimes : \mathbf{Ch}(R)^\bullet \times \mathbf{Ch}(R)^\bullet \to \mathbf{Ch}(R)^\bullet$ on $\mathbf{Ch}(R)^\bullet$. If V, W are two cochain complexes, then we can describe $V \otimes W$ as follows

$$(V \otimes W)^n = \bigoplus_{p+q=n} (V^p \otimes_R W^q)$$

and the differential map is defined to be

$$d(v \otimes w) = dv \otimes w + (-1)^{\deg(v)} v \otimes dw$$

Next we check this map is actually a differential. Suppose $v \otimes w \in V^p \otimes W^q$,

$$\begin{aligned} dd(v \otimes w) &= d\big(dv \otimes w + (-1)^p v \otimes dw\big) \\ &= ddv \otimes w + (-1)^{p+1} dv \otimes dw + (-1)^p \big(dv \otimes dw + (-1)^p v \otimes ddw\big) \\ &= 0 \end{aligned}$$

For two maps $f : V \to V'$, $g : W \to W'$, there will be a chain map $f \otimes g : V \otimes W \to V' \otimes W'$ sending $v \otimes w$ to $f(v) \otimes g(w)$. For simplicity we also use the symbol R to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \cdots$$

with all terms zero except the 0-th position which is R. Then there will exist a canonical identity $V \otimes R = V$. Moreover there is a canonical isomorphism

$$V \otimes W \xrightarrow{\sim} W \otimes V,\ v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v \quad \text{(Koszul sign)}$$

We only need to check this map is a chain map which is obvious from the following diagram

$$\begin{array}{ccc} v \otimes w & \xmapsto{\qquad\qquad\sim\qquad\qquad} & (-1)^{pq} w \otimes v \\ \downarrow d & & \downarrow d \\ dv \otimes w + (-1)^p v \otimes dw & \xmapsto{\sim} & (-1)^{(p+1)q} w \otimes dv + (-1)^p (-1)^{p(q+1)} dw \otimes v \end{array}$$

This tensor product is associative. For $(V \otimes W) \otimes U$ we have

$$\big((V \otimes W) \otimes U\big)^n = \bigoplus_{p+q=n} \big((V \otimes W) \otimes U^q\big) = \bigoplus_{p+q=n} \big(\bigoplus_{s+t=p} (V^s \otimes W^t) \otimes U^q\big) = \bigoplus_{s+t+q=n} V^s \otimes W^t \otimes U^q$$

where the differential is such that

$$\begin{aligned} v \otimes w \otimes u \mapsto \quad & d(v \otimes w) \otimes u + (-1)^{s+t} v \otimes w \otimes du \\ &= dv \otimes w \otimes u + (-1)^s v \otimes dw \otimes u + (-1)^{s+t} v \otimes w \otimes du \end{aligned}$$

And for $V \otimes (W \otimes U)$ its differential is such that

$$\begin{aligned} v \otimes w \otimes u \mapsto \quad & dv \otimes w \otimes u + (-1)^s v \otimes d(w \otimes u) \\ &= dv \otimes w \otimes u + (-1)^s v \otimes dw \otimes u + (-1)^s v \otimes (-1)^t w \otimes du \end{aligned}$$

Therefore $(V \otimes W) \otimes U = V \otimes (W \otimes U)$.

Now let us construct the internal hom funcotr $\mathrm{Hom}_R^\bullet(-,-)$. For any two cochain complexes V and W define

$$\begin{aligned} \mathrm{Hom}_R^n(V, W) &:= \{f : V \to W | R\text{-linear map of graded objects of degree } n\} \\ &= \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(V^p, W^{p+n}) \end{aligned}$$

Note that for $(f^p : V^p \to W^{p+n})_{p\in\mathbb{Z}}$ they may not commutes with the differential. The differential on $\mathrm{Hom}^\bullet(V,W)$ is such that

$$\mathrm{Hom}^n_R(V,W) \to \mathrm{Hom}^{n+1}_R(V,W),\ \ d(f) = d_W \circ f - (-1)^{\deg(f)} f \circ d_V$$

It's actually a differential since

$$\begin{aligned} d(d(f)) &= d\big(d_W f - (-1)^n f d_V\big) = d(d_W f) - (-1)^n d(f d_{V)} \\ &= d_W d_W f - (-1)^{n+1} d_W f d_V - (-1)^n d_W f d_V + (-1)^n (-1)^{n+1} f d_V d_V \\ &= 0 \end{aligned}$$

With this internal hom functor it's clear to see a morphism $f : V \to W$ of graded objects is a cochain map i.e. in $\mathbf{Ch}(R)^\bullet$ if and only if $\deg(f) = 0$ and $d_W f = f d_V$ which is equivalent to $d(f) = 0$ when viewing $f \in \mathrm{Hom}^0_R(V,W)$. Next we prove there are natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Ch}(R)^\bullet}(U \otimes V.W) \cong \mathrm{Hom}_{\mathbf{Ch}(R)^\bullet}(U, \mathrm{Hom}^\bullet_R(V,W))$$

Proof. Given a cochain map $f : U \otimes V \to W$ sending $u^p \otimes v^q$ to w^n where $n = p+q$, then fixing u^p we obtain a set of maps $(f^q_u)_{q\in\mathbb{Z}}$ where $f^q_u : V^q \to W^{p+q}$ of degree p sending v^q to w^n the image of $u^p \otimes v^q$ via the cochain map f. Before checking it's a cochain map $U \to \mathrm{Hom}^\bullet_R(V,W)$, it's obvious to see it's functorial on U and W.

Since f is a cochain map, $f(d(u^p \otimes v^q)) = d(w^n)$ which implies $dw = f_{du}(v) + (-1)^p f_u(dv)$. Then $f_{du}(v) = dw - (-1)^p f_u(dv)$. To see the map $U \to \mathrm{Hom}^\bullet_R(V,W)$ is a cochain map, $d(f_u) = d_W f_u - (-1)^p f_u d_V$ sending v^q to $dw - (-1)^p f_u(dv)$ which is just $f_{du}(v)$. Therefore $f_{du} = d(f_n)$.

On the other hand given a cochain map $f : U \to \mathrm{Hom}^\bullet_R(V,W)$ sending u^p to $(f_u : V \to W)$ of degree p, we can define a map $U \otimes V \to W$ by sending $u^p \otimes v^q$ to $f_u(v^q)$. We check it's a chain map. Since f is a cochain map, $d(f_u) = d_W f_u - (-1)^p f_u d_V = f_{du}$. The map we state above sends $du \otimes v + (-1)^p u \otimes dv$ to $f_{du}(v) + (-1)^p f_u(dv)$ which is just $d_W f_u(v)$.

It's clear to see the above two maps are inverse to each other. □

On $\mathbf{Ch}(R)^{\le 0}$ the internal hom for V and W will change to be the truncation $\tau^{\le 0}\mathrm{Hom}^\bullet_R(V,W)$ whose 0-term is the kernel of the differential. With this construction $\mathbf{Ch}(R)^{\le 0}$ will be a *closed symmetric monoidal model category* [31, Proposition 4.2.13]. But here is another way to define the tensor product on $\mathbf{Ch}(R)^{\le 0}$ as follows,

$$A \otimes B := N(\Gamma(A) \otimes \Gamma(B))$$

which is much different from the first one. But inheriting from $s\mathbf{Mod}_R$, $\mathbf{Ch}(R)^{\le 0}$ will then be a monoidal proper closed simplicial model category [34, Lemma 1.5].

2.4.2 Model Structure for CDGAs

With the standard tensor product the monoid in $\mathbf{Ch}(R)^{\le 0}$ will be a *differential graded algebra* (dga) i.e. a cochain comlex $A^\bullet$ with the unit $1 \in A^0$ and multiplication $A^n \times A^m \to A^{n+m}$ satisfying $d(ab) = (da)b + (-1)^{|a|}a(db)$ where $|a|$ is the degree of a. We can also write it as $A^\bullet = \oplus_{n\le 0} A^n$. Moreover a *commutative differential graded algebra* (cdga) is a graded commutative dga in other words it satisfies $ab = (-1)^{|a||b|}ba$. The category of commutative monoids in $\mathbf{Ch}(R)^{\le 0}$ is denoted by $\mathbf{cdgA}^{\le 0}_R$.

Suppose $\mathbf{Alg}_R$ is the category of commutative R-algebras where R is a commutative ring. Then there is a forgetful functor $U : \mathbf{Alg}_R \to \mathbf{Mod}_R$ whose left adjoint functor is the symmetric algebra functor $\mathrm{Sym}^\bullet(-) : \mathbf{Mod}_R \to \mathbf{Alg}_R$. See Definition 3.1.1 for the definition of symmetric algebras. Then there is an adjoint pair

$$\mathrm{Sym}^\bullet(-) : \mathbf{Mod}_R \rightleftarrows \mathbf{Alg}_R : U$$

There exists a similar adjoint pair between $\mathbf{Ch}(R)^{\le 0}$ and $\mathbf{cdgA}^{\le 0}_R$ which is a bit more complicated. We first generalize the definition of symmetric algebras and exterior algebras for an R-module M in Definition

3.1.1 to the case of a graded R-module $V = \oplus_{i\in\mathbb{Z}} V^i$. The tensor algebra for V does not change which is just $T(V) = \oplus_{n\geq 0} V^{\otimes n}$. But for *graded-symmetric algebra* $\mathcal{S}(V)$ the two sided ideal is generated by elements $a\otimes b-(-1)^{|a||b|}b\otimes a$ and $a\otimes a$ if $\deg(a)$ is odd while the two sided ideal for *exterior algebra* $\bigwedge(V)$ is generated by $a \otimes b + (-1)^{|a||b|} b \otimes a$ and $a \otimes a$ if $\deg(a)$ is even. $\mathcal{S}^n(V)$ and $\bigwedge^n(V)$ are the image of $T^n(V) = V^{\otimes n}$ in $\mathcal{S}(V)$ and $\bigwedge(V)$ respectively. This definition comes from [19, Section A2.3]. In other context the graded-symmetric algebra is also called the *free graded commutative algebra* [30, Section 2.2]. If here R is a $\mathbb{Q}$-algebra, then the condition for $a \otimes a$ can be neglected which is just the definition in [12, Section 1.1].

Important: I'm not really sure whether the definition for the exterior algebra of a graded R-module V where R is not a $\mathbb{Q}$-algebra is correct or not. In [19] there is only the definition for $\mathcal{S}(V)$ and in [12] it's defined on a field of characteristic 0. And I do not find any reference on the case of positive characteristic. But fortunately we actually do not need the definition of the exterior algebra of a graded vector space and we will not use this concept. We will always use the symbol $\bigwedge$ to mean the usual exterior algebra defined in Definition 3.1.1. Moreover readers should be careful that we use different symbols Sym and $\mathcal{S}$ to distinguish the symmetric algebra for a usual module and a graded module. **In the following we will deal with these things on a field k of characteristic 0 for simplicity**. And readers can find that when we use the symbol k we always mean it has characteristic 0 but when we use R it just denotes an arbitrary commutative ring.

Remark 2.4.2. If a graded k-vector spaces V is concentrated in even (resp. odd) degrees, then we will have $\mathcal{S}(V) = \mathrm{Sym}(V)$ (resp. $\mathcal{S}(V) = \bigwedge(V)$). From this if we let $V^{odd} = \oplus_i V^{2i+1}$ and $V^{ev} = \oplus_i V^{2i}$, then $\mathcal{S}(V) = \mathrm{Sym}(V^{ev}) \otimes \bigwedge(V^{odd})$. With the notion of free graded commutative algebras there is also an adjoint pair between the category of graded k-vector spaces and the category of graded commutative k-algebras. Moreover this adjoint pair can be defined between cochain complexes and commutative differential graded algebras (cdgas)

$$\mathcal{S} : \mathbf{Ch}(k)^{\leq 0} \rightleftarrows \mathbf{cdgA}_k^{\leq 0} : U$$

where for a cochain complex V, in degree n the cdga $\mathcal{S}(V)$ has those elements $x_1 \otimes \cdots \otimes x_m$ satisfying $\sum_{i=1}^m \deg(x_i) = n$.

Theorem 2.4.3. *If k is a field of characteristic 0, then $\mathbf{cdgA}_k^{\leq 0}$ admits a model structure such that*

- *weak equivalences are quasi-isomorphisms;*
- *fibrations are those maps inducing surjections for $n < 0$;*
- *cofibrations are those maps having the left lifting property with respect to all trivial fibrations.*

Moreover this model structure is proper.

Sketch of the proof. For $n \leq 0$ we define cochain complexes

$$D(n) := \cdots \longrightarrow 0 \longrightarrow k \longrightarrow k \longrightarrow 0 \longrightarrow \cdots$$

centered at $[n-1, n]$ and

$$S(n) := \cdots \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow \cdots$$

centered at n. Next let $I = \{0 \to S(0),\ i_n : S(n) \to D(n) | n \leq 0\}$ and $J = \{j_n : 0 \to D(n) | n \leq -1\}$. According to [33, Lecture 01], I and J will generate cofibrations and trivial cofibrations respectively via the small object argument (Theorem A.1.11) in $\mathbf{Ch}(k)^{\leq 0}$. Similarly here $\mathcal{S}(I)$ and $\mathcal{S}(J)$ will play the same role. They generate the model structure on $\mathbf{cdgA}_k^{\leq 0}$ and the above adjoint pair will then be a Quillen pair. Here we need Theorem A.4.4 and to prove it satisfies the second condition you will need the assumption of characteristic 0. Moreover cofibrations will be retracts of *semi-free extensions* which are $f : A \to B$ such that $B = A[\{x_i\}]$ is a polynomial extension in an arbitrary number of variables of non-positive degree.

For the left properness you can find a proof in [44, Corollary 3.4]. The right properness is more clear since every object in $\mathbf{cdgA}_k^{\leq 0}$ is fibrant which is dual to [14, Proposition 2.3.27]. □

Remark 2.4.4. In this remark we explain why we need to suppose k is of characteristic 0. The main idea follows [73, Section 5.1]. Assume $\mathrm{char}(k) = p > 0$. We know if $\mathbf{cdgA}_k^{\leq 0}$ admits the model structure in the theorem above, then the adjoint pair $(\mathcal{S}, U)$ will be a Quillen pair and especially $\mathcal{S}$ will preserve cofibrations and trivial cofibrations. Notice that in $\mathbf{Ch}(k)^{\leq 0}$, $D(n)$ is a cofibrant object with trivial homology groups. This means $\mathcal{S}(j_n)$ will be quasi-isomorphisms and $\mathcal{S}(D(n))$ will have the same homology groups as k when viewing it as a complex centered at 0. Now let us consider $\mathcal{S}(D(-1)) = \mathrm{Sym}(k) \otimes \bigwedge(k)$. Let x denote the unit element 1 of k in $\mathrm{Sym}(k)$ at degree -2 whose differential dx will be the unit element 1 of k in $\bigwedge(k)$ at degree -1. Then x^p of degree $-2p$ formally exits with differential $d(x^p) = px^{p-1}dx = 0$ by induction. Since $\mathcal{S}(D(-1))$ is quasi-isomorphic to k, there will exit and element y in $\mathcal{S}(D(-1))$ of degree $-2p-1$ satisfying $d(y) = x^p$. For the reason of degree y must have the factor of dx but then its differential dy will be zero or have some factor dx. All in all dy can not be x^p.

This statement can to applied to any $D(n)$ when n is odd. Just note that our notation $D(n)$ here is a bit different from other references where n should be supposed to be even.

Remark 2.4.5. There are model structures on $\mathbf{cdgA}_k$ and $\mathbf{cdgA}_k^{\geq 0}$ as well. For $\mathbf{cdgA}_k$,

- weak equivalences are quasi-isomorphisms;
- fibrations are those maps inducing surjection for all $n \in \mathbb{Z}$;
- cofibrations are those maps having the LLP wrt all trivial fibrations.

As for $\mathbf{cdgA}_k^{\geq 0}$,

- weak equivalences are quasi-isomorphisms;
- fibrations are those maps inducing surjection for $n \geq 0$;
- cofibrations are those maps having the LLP wrt all trivial fibrations.

Proofs for them are all similar. You can find them in [9, §4.] and [30, Chapter 4].

The theory of commutative algebras is actually contained in the homotopy theory of cdgas. For a commutative algebra B we can view it as a cdga centered at 0 which induces a functor $\mathbf{Alg}_k \to \mathbf{cdgA}_k^{\leq 0}$ sending isomorphisms to quasi-isomorphisms. On the other hand for any cdga A we can associate it with the commutative algebra H^0A. This functor $H^0 : \mathbf{cdgA}_k^{\leq 0} \to \mathbf{Alg}_k$ sends quasi-isomorphisms to isomorphisms. Therefore the two functors can defined on homotopy categories. Moreover they form an adjoint pair

$$\mathrm{Hom}_{\mathbf{Alg}_k}(H^0(A), B) \cong \mathrm{Hom}_{\mathbf{cdgA}_k^{\leq 0}}(A, B)$$

Proposition 2.4.6. *The induced functor* $\mathbf{Alg}_k \to \mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})$ *is fully faithful.*

Proof. Suppose $A \in \mathbf{cdgA}_k^{\leq 0}$, $B \in \mathbf{Alg}_k$ and QA is a cofibrant replacement in $\mathbf{cdgA}_k^{\leq 0}$. Then

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})}(A, B) = \mathrm{Hom}_{\mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})}(QA, B)$$

Since any object in $\mathbf{cdgA}_k^{\leq 0}$ is fibrant, from Proposition A.2.18 we see

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})}(QA, B) = [QA, B] = \mathrm{Hom}_{\mathbf{cdgA}_k^{\leq 0}}(QA, B)/\sim$$

Notice that the diagonal map $B \to B \times B$ is a fibration since there is nothing on $n < 0$. Therefore it's a path object for B (Definition A.2.5). Then from the definition of right homotopy (Definition A.2.6), we have

$$\mathrm{Hom}_{\mathbf{cdgA}_k^{\leq 0}}(QA, B)/\sim = \mathrm{Hom}_{\mathbf{cdgA}_k^{\leq 0}}(QA, B) = \mathrm{Hom}_{\mathbf{Alg}_k}(H^0(QA), B) = \mathrm{Hom}_{\mathbf{Alg}_k}(QA, B)$$

The fully faithfulness is clear if we suppose A is a commutative algebra. □

In the following we continue to explore structures on $\mathbf{cdgA}_k^{\leq 0}$ or $\mathbf{cdgA}_k$ further.

Definition 2.4.7. For any non-negative integer n, the *algebra of polynomial differential forms* $\Omega_n^\bullet = \Omega^\bullet(\Delta^n)$ on the algebraic n-simplex is the cdga

$$\Omega_n^\bullet := k[t_0, \cdots, t_n, dt_0, \cdots, dt_n]/(\sum t_i - 1, \sum dt_i)$$

where t_i's are at degree 0, dt_i's are at degree 1 and the product is the wedge product.

Remark 2.4.8. $\Omega_n^0 = k[t_0, \cdots, t_n]/(\sum t_i - 1) \subseteq \mathbb{A}_k^{n+1}$ is actually the standard n-simplex in the affine space $\mathbb{A}_k^{n+1}$ of dimension $n+1$ and $\Omega_0^\bullet = k[t]/(t-1, dt) = k$ will be the unit cdga. For $q > p$ we will have $\Omega_p^q = 0$.

Given a map $f : [n] \to [m]$ in Δ, it will induce a map between cdgas

$$\Omega^\bullet(f) : \Omega_m^\bullet \to \Omega_n^\bullet, \;\; t_i \mapsto \sum_{f(j)=i} t_j$$

Therefore we can regard $\Omega^\bullet$ as a simplicial object in $\mathbf{cdgA}_k$. Next from the Kan extension [14, Theorem 1.1.10], this functor can be extended to be defined on $s\mathbf{Set}$.

$$\Omega^\bullet : s\mathbf{Set} \to (\mathbf{cdgA}_k)^{op}, \;\; X \mapsto \operatorname*{colim}_{\Delta^n \to X} \Omega_n^\bullet$$

which admits the right adjoint functor.

Definition 2.4.9. Fixing any non-negative integer q, Ω^q will be a simplicial set sending any $[p]$ to Ω_p^q. The *piecewise linear de Rham funcor* or simply *PL de Rham functor* is defined to be

$$\Omega_{\mathrm{PLdR}} : s\mathbf{Set} \to (\mathbf{cdgA}_k^{\geq 0})^{op}, \;\; X \mapsto \mathrm{Hom}^\bullet_{s\mathbf{Set}}(X, \Omega)$$

where $\mathrm{Hom}^q_{s\mathbf{Set}}(X, \Omega) = \mathrm{Hom}_{s\mathbf{Set}}(X, \Omega^q)$ and the differential sends $f : X \to \Omega^q$ to $d_\Omega \circ f$.

Proposition 2.4.10. *The PL de Rham functor Ω_{PLdR} admits the right Quillen adjoint functor*

$$F : (\mathbf{cdgA}_k^{\geq 0})^{op} \to s\mathbf{Set}, \;\; C \mapsto \big([p] \mapsto \mathrm{Hom}_{\mathbf{cdgA}_k^{\geq 0}}(C, \Omega_p^\bullet)\big)$$

Proof. See [30, Theorem 6.2] or [9, §8.]. □

Since $\Omega^\bullet$ and $\Omega^\bullet_{\mathrm{PLdR}}$ have the same right adjoint functor, they are isomorphic to each other. This adjunction is useful in rational homotopy theory.

Remark 2.4.11. In this remark we describe the simplicial structure on $\mathbf{cdgA}_k$. For any two cdgas A and B we define the internal simplicial set $\mathrm{Map}(A, B) \in s\mathbf{Set}$ such that

$$\mathrm{Map}(A, B)_n := \mathrm{Hom}_{\mathbf{cdgA}_k}(A, \Omega_n^\bullet \otimes_k B)$$

Clearly $\mathrm{Map}(A, B)_0 = \mathrm{Hom}_{\mathbf{cdgA}_k}(A, B)$. The composition is defined via the following natural isomorphism [9, 5.1]

$$\mathrm{Hom}_{\mathbf{cdgA}}(A, \Omega_n^\bullet \otimes_k B) \cong \mathrm{Hom}_{\Omega_n^\bullet}(\Omega_n^\bullet \otimes_k A, \Omega_n^\bullet \otimes_k B)$$

where the right side means maps preserving the left dg $\Omega_n^\bullet$-module structure. This generalizes the classical result for tensor products of modules. [9, Propsition 5.3] shows this nearly defines a simplicial model structure on $\mathbf{cdgA}_k$. As for the internal hom functor for $\mathbf{cdgA}_k^{\leq 0}$, we can only consider its truncation $\tau^{\leq 0}$ of $\Omega_n^\bullet \otimes_k B$ which means

$$\mathrm{Map}_{\mathbf{cdgA}_k^{\leq 0}}(A, B)_n = \mathrm{Hom}_{\mathbf{cdgA}_k^{\leq 0}}(A, \tau^{\leq 0}(\Omega_n^\bullet \otimes_k B))$$

This nearly simplicial model structure on $\mathbf{cdgA}_k$ actually defines a correct derived internal hom functor $\mathbb{R}\mathrm{Map}(A, B) = \mathrm{Map}(QA, \Omega_n^\bullet \otimes_k B)$ in the sense that the simplicial cdga $\Omega^\bullet \otimes_k B$ is a simplicial resolution of B (Definition A.4.19) so that we can obtain a good simplicial localization theory [18]. For the proof that $B \to \Omega^\bullet \otimes_k B$ is a Reedy fibrant replacement in $s\mathbf{cdgA}_k$ you can consult with [2, Proposition 4.5].

2.4.3 Cotangent Complexes

If we want to do algebra or geometry on cdgas, we need to introduce the concept of *cotangent complex* first which in Toën's opinion is the true origin of derived algebraic geometry [64]. We first introduce the classical one and then talk about a version for cdgas.

In a category $\mathcal{C}$, an *abelian object* A is an object in $\mathcal{C}$ satisfying categorical axioms similar to abelian groups which is equivalent to that the representable functor $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to \mathbf{Set}$ factors through $\mathbf{Ab}$ the category of abelian groups. We use $\mathcal{C}_{ab}$ to denote the full subcategory of $\mathcal{C}$ consisting of abelian objects. For example $s\mathbf{Set}_{ab} = s\mathbf{Ab}$ the category of simpliical abelian groups. If both $\mathcal{C}$ and $\mathcal{C}_{ab}$ have model structures and the abelianization functor $\mathcal{C} \to \mathcal{C}_{ab}$ is left Quillen, then we will be interested in its left derived functor.

Next we fix a k-algebra A and consider abelian objects in $\mathbf{Alg}_k/A = \mathbf{Alg}_k \downarrow A$ the category of k-algebras over A.

Definition 2.4.12. Let M be an A-module. Then the *trivial extension* of A along M is defined to be a k-algebra $A \oplus M$ such that

$$(a, m) \cdot (b, n) = (ab, an + bm)$$

We denote this k-algebra by $A \ltimes M$. With the projection $pr_1 : A \rtimes M \to A$, we can view it as a k-algebra over A.

Proposition 2.4.13. *$(\mathbf{Alg}_k/A)_{ab}$ is equivalent to the category of all trivial extensions of A.*

Proof. See [51, Proposition 1.4] or [15, Proposition 1.15]. □

And therefore $(\mathbf{Alg}_k/A)_{ab}$ will be equivalent to $\mathbf{Mod}_A$ the category of A-modules. Next let $f : B \to A \ltimes M$ be a map between k-algebras over A. Suppose $f(b) = (f_1(b), f_2(b))$. Since f is over A, f_1 is determined by the structure of B over A which means a map $B \to A$ of k-algebras. Therefore f is then determined by the map f_2. Since f should be a map of algebras, $f(bb') = f(b)f(b')$ and this is equivalent to that $f_2(bb') = bf_2(b') + b'f_2(b)$ where the multiplication is via the map $B \to A$. This then means f_2 is k-linear derivation $B \to M$ and we obtain isomorphisms

$$\mathrm{Hom}_{\mathbf{Alg}_k/A}(B, A \ltimes M) \cong \mathrm{Der}_k(B, M) \cong \mathrm{Hom}_B(\Omega^1_{B/k}, M) \cong \mathrm{Hom}_A(\Omega^1_{B/k} \otimes_B A, M)$$

which means there exists an adjoint pair

$$\Omega^1_{-/k} \otimes_{-} A : \mathbf{Alg}_k/A \rightleftarrows \mathbf{Mod}_A : A \ltimes -$$

What we talked above can be extended to the simplicial case obviously and we have an adjoint pair

$$\Omega^1_{-/k} \otimes_{-} A : s\mathbf{Alg}_k/A \rightleftarrows s\mathbf{Mod}_A : A \ltimes -$$

where the two functors are applied objectwise and A is a simplicial k-algebra.

From Theorem 2.4.1 $s\mathbf{Ab}$, $s\mathbf{Alg}_k$ and $s\mathbf{Mod}_k$ are all simplicial model categories. Since on underlying simplicial sets $A \ltimes -$ is just $A \times -$, the functor $A \rtimes -$ will preserve Kan fibrations and trivial Kan fibrations. Thus the adjoint pair above is a Quillen pair and we obtain derived functors

$$\mathbb{L}(\Omega^1_{-/k} \otimes_{-} A) : \mathrm{Ho}(s\mathbf{Alg}_k/A) \rightleftarrows \mathrm{Ho}(s\mathbf{Mod}_A) : \mathbb{R}(A \ltimes -)$$

Moreover $\mathbb{L}(\Omega^1_{-/k} \otimes_{-} A)(B) = \Omega^1_{QB/k} \otimes_{QB} A$ where QB is a cofibrant replacement of B in $s\mathbf{Alg}_k/.A$ which is actually the usual cofibrant replacement in $s\mathbf{Alg}_k$.

Definition 2.4.14. For a commutative k-algebra A, the *cotangent complex* $\mathbb{L}_{A/k}$ is defined to be $\mathbb{L}(\Omega^1_{-/k} \otimes_- A)(A)$ which is $\Omega^1_{QA/k} \otimes_{QA} A$ in $\mathrm{Ho}(s\mathbf{Mod}_A)$. And the *André-Quillen homology* of A is defined to be $D_n(A/k) = \pi_n \mathbb{L}_{A/k}$ the homotopy group of cotangent complex.

For any A-module M we define André-Quillen homology and André-Quillen cohomology with coefficient M as

$$D_n(A/k, M) := H_n(N(\mathbb{L}_{A/k}) \otimes_A M) \quad \text{and} \quad D^n(A/k; M) := H^n(\mathrm{Hom}_A(N(\mathbb{L}_{A/k}), M))$$

respectively.

Actually via Dold-Kan correspondence we can change every simplicial A-module into a chain complex of A-modules and compute its homology groups which will be isomorphic to homotopy groups of the original simplicial module. Classical theory for cotangent complex via free simplicial resolution computes that $D_0(A/k) = \Omega^1_{A/k}$. Here we only talk about the affine case. To obtain the definition of cotangent complex for schemes, these local (affine) constructions should be glued together by the use of *standard simplicial resolution*. Details can be found in [32]. Finally we have the following theorem to characterize smooth morphisms and étale morphisms.

Theorem 2.4.15. *Suppose the map $f : X \to Y$ of schemes is locally of finite presentation.*

(1). f is smooth if and only if the map $\mathbb{L}_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism and $\Omega^1_{X/Y}$ is locally free of finite rank where we also view $\Omega^1_{X/Y}$ as a complex centered at 0.

(2). f is étale if and only if $\mathbb{L}_{X/Y}$ is isomorphic to 0 in the derived category.

Proof. A statement for affine case can be found in [53, Theoren 5.4 and Theorem 5.5]. And proofs can be found in [51, Proposition 5.3] or [32, Proposition 3.1.1 and Proposition 3.1.2]. □

Arguments above can be applied without change to the case of cdgas thanks to Dold-Kan correspondence. For a cdga A the category of (unbounded) differential graded A-modules is denoted by $\mathbf{dgMod}_A$ where the multiplication and differential should satisfy $d(ax) = (da)x + (-1)^{|a|} adx$. And the category of (unbounded) differential graded A-algebras is denoted by $\mathbf{cdgA}_A$ which is equivalent to $A/\mathbf{cdgA}_k = A \downarrow \mathbf{cdgA}_k$. Let M be an object in $\mathbf{dgMod}_A$ we define a new cdga the *trivial extension* $A \ltimes M$ such that its underlying cochain complex is $A \oplus M$ and the multiplication is induced by the usual trivial extension in Definition 2.4.12. These definitions or constructions are also valid for $\mathbf{cdgA}_k^{\leq 0}$. Then we have a Quillen pair

$$\Omega^1_{-/k} \otimes_- A : \mathbf{cdgA}_k^{\leq 0}/A \rightleftarrows \mathbf{dgMod}_A^{\leq 0} : A \ltimes -$$

The *cotangent complex* for a cdga A is thus defined to be

$$\mathbb{L}_{A/k} := \mathbb{L}(\Omega^1_{-/k} \otimes_- A)(A) = \Omega^1_{QA/k} \otimes_{QA} A$$

There is a more general case where we consider cdgas over a cdga. Suppose $B \to A$ is a map of cdgas. Then A is an object in $\mathbf{cdgA}_B^{\leq 0}$. We consider the cofibrant replacement QA of A in $\mathbf{cdgA}_B^{\leq 0}$ which means the following factorization

$$B \rightarrowtail QA \xrightarrow{\sim} A$$

where $B \to QA$ is a cofibration. Then the *relative cotangent complex* is defined to be $\mathbb{L}_{A/B} := \Omega^1_{QA/B} \otimes_{QA} A$ which is an object in $\mathrm{Ho}(\mathbf{dgMod}_A^{\leq 0})$.

Given a map of cdgas $R \to A$ in $\mathbf{cdgA}_k^{\leq 0}$, $\Omega^1_{A/R}$ is the module of Kähler differentials in the usual sense but it can be an object in $\mathbf{dgMod}_A^{\leq 0}$. Note that $\Omega^1_{A/R}$ is isomorphic to I/I^2 where $I = \ker(A \otimes_R A \to A)$. From this point the differential graded structure is clear. This construction has a universal property like the classical one but here for an R-derivation $d : A \to M$ from A to an object M in $\mathbf{dgMod}_A$, it should satisfy the graded Leibniz rule

$$d(a_1 \cdot a_2) = d(a_1) \cdot a_2 + (-1)^{|a_1|} a_1 \cdot d(a_2)$$

Then we have a natual isomorphism

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega^1_{A/R}, M)$$

2.5 Derived Geometric Stacks

In this section we talk about the geometry on $\mathbf{cdgA}_k^{\leq 0}$ and we use the symbol $\mathbf{dAff}_k$ to denote the opposite category of $\mathbf{cdgA}_k^{\leq 0}$. For any object A of $\mathbf{cdgA}_k^{\leq 0}$, $\mathrm{Spec}A$ will denote the same object in $\mathbf{dAff}_k$ which is called *affine derived scheme*. Now we let $\mathcal{C} = \mathbf{dAff}_k$ and apply methods in the first two sections to $\mathcal{C}$. Objects in the essential image of $\mathrm{Ho}(\mathbf{dAff}_k^{\wedge})$ in $\mathrm{Ho}(sPr(\mathbf{dAff}_k))$ are called *derived prestacks*. Our first task here will be to equip $\mathbf{dAff}_k$ with a model pretopology

Definition 2.5.1. [14] A morphism $f : A \to B$ in $\mathbf{cdgA}_k^{\leq 0}$ is *flat* (resp. *smooth, étale, a Zariski opem immersion*) if

(1). for any $n < 0$ the natural morphism

$$H^n(A) \otimes_{H^0(A)} H^0(B) \longrightarrow H^n(B)$$

is an isomorphism;

(2). the induced morphism of affine schemes $\mathrm{Spec}H^0(B) \to \mathrm{Spec}H^0(A)$ is flat (resp. smooth, étale, a Zariski open immersion).

Fact 2.5.2. For an object A of $\mathbf{cdgA}_k^{\leq 0}$, $H^n(A)$ will naturally have a $H^0(A)$-module structure. Suppose $a + dA^{-1}$ and $x + dA^{n-1}$ are elements of $H^0(A)$ and $H^n(A)$ respectively. Then $(a + dA^{-1}) \cdot (x + dA^{n-1}) := ax \in H^n(A)$. This multiplication is well defined since for any $b \in A^{-1}$ and $y \in A^{n-1}$, xdb and ady will be a coboundaries. This is clear when we notice $d(xb) = (-1)^{|x|} xdb$ and $d(ay) = ady$.

Remark 2.5.3. In [68] or [49] there are very abstract definitions or characterizations for maps we defined above and many of them are dependent on cotangent complexes [68, Definition 1.2.6.1]. Note that in [68] a morphism is a Zariski open immersion (resp. smooth, étale) if it's a formal Zariski open immersion (resp. formally smooth, formally étale) and of finite presentation. Moreover for a map of commutative algebras if we view it as a morphism between cdgas, it will be stronger to say it's finitely presented as cdgas than as a map of commutative algebras. However our definitions here will be equivalent to those abstract ones which is shown in [68, Proposition 2.2.2.5 and Theorem 2.2.2.6].

These equivalences are very interesting since they provide us a homotopical perspective to consider geometric morphisms in algebraic geometry. For example they tell us that for a finitely presented map $A \to B$ of commutative algebras, $\mathrm{Spec}B \to \mathrm{Spec}A$ is a Zariski open immersion if and only if the induced restriction functor $D^{\leq 0}(B) \to D^{\leq 0}(A)$ between derived categories is fully faithful. A direct proof for this statement can be found in the post "A derived characterization of open immersions" written by Akhil Mathew.

Definition 2.5.4. A family of morphisms $\{\mathrm{Spec}A_i \to \mathrm{Spec}A | i \in I\}$ in $\mathbf{dAff}_k$ is a *Zariski covering* (resp. *étale covering, flat covering*) if and only if every map $A \to A_i$ is a Zariski open immersion (resp. étale, flat) and there exists a finite subset $J \subseteq I$ such that

$$\coprod_{j \in J} \mathrm{Spec}H^0(A_i) \to \mathrm{Spec}H^0(A)$$

is a Zariski (resp. étale, flat) covering in the usual sense.

[14] [62, Definition 4.1]

This definition will define Zariski (resp. étale, fppf) model pretopology. Although here Zariski topology is enough for our purpose, we insist using étale topology. In the following if we do not remind specially $\mathbf{dAff}_k$ is always equipped with étale model pretopology.

Remark 2.5.5. In this remark we check the notion of étale covering actually defines étale model pretopology in the sense of Definition 2.2.5. The "stability" condition is trivially satisfied since for any isomorphism in $\mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})$ we can find a quasi-isomorphism in its preimage in $\mathbf{cdgA}_k^{\leq 0}$. The definition for étale covering is on the homology level and the "stability" condition follows from that any isomophism of commutative algebras is a usual étale covering.

To prove the "composition" condition we also first lift maps in $\mathrm{Ho}(\mathbf{cdgA}_k^{\leq 0})$ to $\mathbf{cdgA}_k^{\leq 0}$. Suppose $\{\mathrm{Spec}A_i \to \mathrm{Spec}A | i \in I\}$ is a covering and for any $i \in I$, $\{\mathrm{Spec}A_{ij} \to \mathrm{Spec}A_i | j \in J_i\}$ is a covering. Then for $n < 0$,

$$\begin{aligned} H^n(A_{ij}) &\cong H^n(A_i) \otimes_{H^0(A_i)} H^0(A_{ij}) \\ &\cong H^n(A) \otimes_{H^0(A)} H^0(A_i) \otimes_{H^0(A_i)} H^0(A_{ij}) \\ &\cong H^n(A) \otimes_{H^0(A)} H^0(A_{ij}) \end{aligned}$$

and at the homology level of degree 0 we will obtain the usual étale covering from.

To prove the "homotopy base change" condition, we suppose $\{\mathrm{Spec}A_i \to \mathrm{Spec}A | i \in I\}$ is a covering and $\mathrm{Spec}B \to \mathrm{Spec}A$ is a map in $\mathbf{cdgA}_k^{\leq 0}$. We know $\mathbf{cdgA}_k^{\leq 0}$ is proper. Therefore from Theorem A.4.10, to compute the corresponding homotopy pullback we only need to compute the pullback of $\mathrm{Spec}C \to \mathrm{Spec}A \leftarrow \mathrm{Spec}A_i$ where $A \to B$ factors as $A \to C \to B$ the composition of a trivial fibration with a cofibration. Note that

$$\mathrm{Spec}C \times_{\mathrm{Spec}A} \mathrm{Spec}A_i = \mathrm{Spec}(C \otimes_A A_i)$$

A direct computation implies

$$H^0(C \otimes_A A_i) \cong H^0(C) \otimes_{H^0(A)} H^0(A_i)$$

Then it's clear

$$H^n(C) \otimes_{H^0(C)} H^0(C \otimes_A A_i) \cong H^n(C) \otimes_{H^0(A)} H^0(A_i)$$

Next we need to compute $H^n(C \otimes_A A_i)$. Classical Künneth spectral sequence [57, Theorem 10.90] tells us that if A is just a commutative algebra R, one of C and A_i are flat over R, then there is a spectral sequence such that

$$E_2^{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}_p^R(H^s(C), H^t(A_i)) \overset{p}{\Longrightarrow} H^{p+q}(C \otimes_R A_i)$$

There is a generalization for cdgas. In [39, Construction 7.2.1.18], Lurie shows there is a spectral sequence such that

$$E_2^{p,q} = \mathrm{Tor}_p^{H^*A}(H^*(C), H^*(A_i))_q \overset{p}{\Longrightarrow} H^{p+q}(C \otimes_A A_i)$$

where H^*A is a graded commutative algebra while H^*C and H^*A_i are graded H^*A-modules. Since we suppose A_i is étale over A, H^0A_i will be étale over H^0A in the usual sense and especially it's flat. And moreover we have $H^*A_i \cong H^*A \otimes_{H^0A} H^0A_i$. H^*A_i is then étale especially flat over H^*A. Therefore the spectral sequence above gives

$$\mathrm{Tor}_0^{H^*A}(H^*(C), H^*(A_i)) = H^*C \otimes_{H^*A} H^*A_i = H^*C \otimes_{H^0A} H^0A_i \cong H^*(C \otimes_A A_i)$$

The statement above is Proposition 7.2.2.13 in [39].

Finally since usual étale coverings are stable under pullbacks, we complete the proof.

Remark 2.5.6. Applying techniques in previous sections to $\mathbf{dAff}_k$ with étale model pretopology, we obtain the model category $\mathbf{dAff}_k^{\sim}$ of derived stacks whose homotopy theory with respect to étale π_*-equivalences is denoted by $\mathbf{dSt}_k$.

From Theorem 2.2.3 via the model Yoneda embedding we can view an affine derived scheme $\mathrm{Spec}A$ as a derived prestack. Here we let

$$\underline{\mathrm{Spec}} = \underline{h} : \mathbf{dAff}_k \to sPr(\mathbf{dAff}_k), \ \ \mathrm{Spec}A \mapsto \mathrm{Hom}_{\mathbf{dAff}_k}(\Gamma(-), \mathrm{Spec}A)$$

where the simplicial resolution functor Γ has a concrete description. According to Remark 2.4.11 we can let

$$\Gamma(B) = \tau^{\leq 0}(\Omega^\bullet \otimes_k B)$$

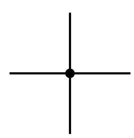
Then $\mathrm{Spec}A$ means it's an object in $\mathbf{dAff}_k$ while $\underline{\mathrm{Spec}}A$ means it's in $\mathbf{dAff}_k^\wedge$ and $\mathbb{R}\underline{\mathrm{Spec}}A$ is in $\mathrm{Ho}(\mathbf{dAff}_k^\wedge)$. Moreover [68, Lemma 1.3.2.5] shows a model pretopology τ satisfying [68, Assumption 1.3.2.2] is subcanonical in the sense that every *representable object* is a stack. Therefore the derived prestack $\mathbb{R}\underline{\mathrm{Spec}}A$ is actually a derived stack. For any object in $\mathbf{dSt}_k$ isomorphic to some affine derived scheme, we also call it an affine derived scheme.

Why we need derived algebraic geometry? Or compared with classical algebraic geometry what's the advantage of derived algebraic geometry? Actually the most important one I think is to help us deal with some singular situations such as some bad intersections. We give some basic examples here.

Example 2.5.7. This example comes from [11, Section 1.1].

At first we need to know which intersections are good and which ones are bad. For good intersections we mean two smooth subvarieties meet *transversely* inside an ambient smooth variety which means tangent spaces at the intersection point of the two subvarieties generate the whole tangent space of the ambient space. In this case this intersection is itself a smooth subvariety whose codimension is the sum of the codimensions of the two subvarieties and satisfies some good properties. This statement is true no matter in differential geometry or algebraic geometry. It's well known that the category of smooth manifolds is not really good since the pullback may not exist. But in the case where two maps are transverse, we can equip the pullback at the set level with a smooth structure as a smooth submanifold.

Next we talk about some easy examples.

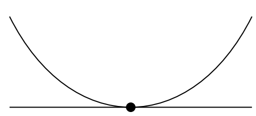


The above intersection is given by $\{x = 0\} \cap \{y = 0\} \subseteq \mathbb{A}_k^2$ which is algebraically defined as

$$k[x, y]/(x) \otimes_{k[x,y]} k[x, y]/(y) = k[x, y]/(x, y) = k$$

with (Krull) dimension 0. And its codimension is 2 the sum of that of two affine lines.

A bad one is given by $\{y = x^2\} \cap \{y = 0\} \subseteq \mathbb{A}_k^2$.



Algebraically it's

$$k[x, y]/(y - x^2) \otimes_{k[x,y]} k[x, y]/(y) = k[x, y]/(y - x^2, y) = k[x]/(x^2)$$

Clearly $\mathrm{Spec}k[x]/(x^2)$ is of (Krull) dimension 0 but is not smooth at the original point and this intersection multiplicity is 2 since as a k-vector space it's of dimension 2. In general the *intersection multiplicity* at an intersection point is used to express how many times the two subvarieties meet at this point. It will be one if the two subvarieties are smooth and intersect transversely [20, Lemma 1.26] but other conditions may be complicated. There is actually a formula due to Serre to compute intersection multiplicities in such complicated cases [20, Theorem 2.7].

A much worse example is about self-intersection. For example $\{y=0\}\cap\{y=0\}=\{y=0\}$ is given by $k[x,y]/(y)=k[x]$ of (Krull) dimension 1 instead of 0. Here how can derived algebraic geometry help us? In derived algebraic geonmetry intersections are computed as homotopy pullbacks. Therefore we need to compute $k[x]\otimes^{\mathbb{L}}_{k[x,y]}k[x]$. Since $\mathbf{cdgA}_k^{\leq 0}$ is proper, we only need to factor one side $k[x,y]\to k[x]$ as a composition of a trivial fibration with a cofibration.

Let $\widetilde{k[x]}=k[x,y,\xi]$ where ξ is a free generator in the degree -1 and $x,\ y$ are in the degree 0. For differential $d\xi=y$ and $dx=dy=0$. Clearly $k[x,y]\to\widetilde{k[x]}$ is semi-free extension hence a cofibration in $\mathbf{cdgA}_k^{\leq 0}$. Next we prove $\widetilde{k[x]}\to k[x]$ sending $\xi,\ y$ to 0 is a trivial fibration. Actually we only need to prove cohomology groups of $\widetilde{k[x]}$ for $n<-1$ are all trivial. An element in $\widetilde{k[x]}^n$ has the form $f(x,y)\xi^n$ where we suppose $n>1$ and compute the $-n$ cohomology group. Since $df(x,y)\xi^n=nf(x,y)y\xi^{n-1}$, $df(x,y)\xi^n=0$ if and only if $f(x,y)=0$. Then it's clear the cohomology group is trivial. Therefore the derived intersection is computed as $\widetilde{k[x]}\otimes_{k[x,y]}k[x]$ where

$$(\widetilde{k[x]}\otimes_{k[x,y]}k[x])^{-n}=k[x,y]\cdot\xi^n\otimes_{k[x,y]}k[x]=k[x]\cdot\xi^n$$

Then $\widetilde{k[x]}\otimes_{k[x,y]}k[x]=k[x,\xi]$ where ξ is at degree -1 and x is at degree 0 with $d\xi=0,\ dx=0$. Here the *virtual dimension* is computed as the difference between the number of even and odd generators which is just 0. And in this example the classical intersection $k[x]$ is just the truncation of the derived one.

For those good intersections we can obtain the same result. We compute the derived intersection of the first example which will be

$$k[x]\otimes^{\mathbb{L}}_{k[x,y]}k[y]=\widetilde{k[x]}\otimes_{k[x,y]}k[y]=k[y,\xi]$$

where $d\xi=y$. This result is quasi-isomorphic to k.

And the derived intersection of the second example is given by

$$k[x]\otimes^{\mathbb{L}}_{k[x,y]}k[x,y]/(y-x^2)=k[x,\xi]$$

where $d\xi=x^2$. Notice that $H^0(k[x,\xi])=k[x]/(x^2)$ is just the non-derived intersection.

Examples above show that classical constructions in algebraic geometry are truncated ones in the derived world. No matter how singular this construction is, in the derived world it seems to be "smooth". This is just the famous **hidden smoothness** principle. There is a more complicated example in [66, Section 1.2]. In Example 2.3.8 we have said if X is a smooth projective curve of genus g, then the moduli stack $\mathscr{B}un_X^n$ is smooth of dimension $n^2(g-1)$. But if X is a smooth projective surface, then this algebraic stack will not be smooth. [66, Section 5.2] shows how we can obtain the derived stack $\mathbb{R}\mathscr{B}un_X^n$ which is smooth in the derived world and the Euler character of its tangent space which is not just a space but a complex, is locally constant.

Now let us consider how can we obtain the concept of *derived scheme* from affine derived schemes. Schemes are obtained by the gluing of affine schemes i.e. they admit a Zariski atlas. This statement has a version for functors which says any Zariski sheaf $\mathbf{Sch}_k^{op}\to\mathbf{Set}$ admitting an open covering by representable subfunctors is itself representable [60, Proposition 2.16]. Our construction here is similar to the classical one.

Definition 2.5.8. [15]

(1). Let $\mathbb{R}\underline{\mathrm{Spec}}A$ be an affine derived scheme and $\mathcal{F}$ a derived stack. A morphism $i:\mathcal{F}\to\mathbb{R}\underline{\mathrm{Spec}}A$ is a *Zariski open immersion* if it satisfies the following conditions.

(a). The map i is a monomorphism.

[15] [49, Definition 4.2] or [63, Definition 3.1.1]

(b). There is a family $\{\mathbb{R}\underline{\mathrm{Spec}}A_i \to \mathcal{F}\}$ of morphisms in $\mathbf{dSt}_k$ such that the induced map

$$\coprod_i \mathbb{R}\underline{\mathrm{Spec}}A_i \to \mathcal{F}$$

is an epimorphism and each composition

$$\mathbb{R}\underline{\mathrm{Spec}}A_i \to \mathcal{F} \to \mathbb{R}\underline{\mathrm{Spec}}A$$

is a Zariski open immersion of cdgas.

(2). A morphism $\mathcal{F} \to \mathcal{G}$ of derived stacks is a *Zariski open immersion* if for any morphism $\mathbb{R}\underline{\mathrm{Spec}}A \to \mathcal{G}$, the induce morphism

$$\mathcal{F} \times^h_{\mathcal{G}} \mathbb{R}\underline{\mathrm{Spec}}A \to \mathbb{R}\underline{\mathrm{Spec}}A$$

is a Zariski open immersion in the above sense.

(3). A derived stack $\mathcal{F}$ is a *derived scheme* if there exists a family $\{\mathbb{R}\underline{\mathrm{Spec}}A_i \to \mathcal{F} | i \in I\}$ of Zariski open immersions such that the induced map

$$\coprod_i \mathbb{R}\underline{\mathrm{Spec}}A_i \to \mathcal{F}$$

is an epimorphism and such a family is called a *Zariski atlas* for $\mathcal{F}$.

For *derived algebraic stacks* or *derived Artin stacks*, we view derived schemes as derived 0-algebraic stacks and do induction based on Definition 2.3.10 where the class $\mathbf{P}$ is chosen to be smooth morphisms. And then we obtain the notion of *derived n-algebraic stacks*. Similarly we have the concept of *derived Deligne-Mumford stacks* as well where $\mathbf{P}$ is chosen to be étale maps. Note that it's a bit different from derived algebraic n-stacks as we have talked before. Also note that in [68] the notion of *derived geometric stacks* is a bit different from derived Artin stacks we said here. But we do not distinguish them. For details you can see Remark 2.3.11.

Like classical Artin stacks there is a criterion for derived Artin stacks named *Artin-Lurie representability theorem* which originally appears in Lurie's phd thesis [37, Theorem 7.1.6]. You can also find some details in [49, Section 5] or [68, Appendix C].

Remark 2.5.9. There is a definition for *dg-schemes* in [13] which says a dg-scheme consists of a pair $X = (X^0, \mathcal{O}^\bullet_X)$ where X^0 is an ordinary scheme and $\mathcal{O}^\bullet_X$ is a sheaf of cdgas on X^0 such that $\mathcal{O}^0_X = \mathcal{O}_{X^0}$ and every $\mathcal{O}^i_X$ is quasi-coherent over $\mathcal{O}_{X^0}$. Here we have $H^0(\mathcal{O}^\bullet_X)$ a quasi-coherent sheaf of algebras and we can obtain a truncation scheme $\pi_0(X) := \mathrm{Spec}_{X^0} H^0(\mathcal{O}^\bullet_X)$ [8, Section 7.1, p288] which is actually a closed subscheme of X^0.

A morphism $f : X \to Y$ between dg-schemes is a quasi-isormophism if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of schemes and $H^n(f) : H^n(\mathcal{O}^\bullet_X) \to H^n(\mathcal{O}^\bullet_Y)$ is an isomorphism of quasi-coherent sheaves for every n. From this definition if we replace X^0 by any open subscheme containing $\pi_0(X)$, then we will obtain a quasi-isomorphic dg-scheme. In this case it seems the right definition for a dg-scheme is actually in the localization category (with respect to quasi-isomorphisms) which unfortunately is difficult to describe and work with. But if we only deal with the real category of dg-schemes, it will be a bit restrictive since the gluing process here is strict not up to quasi-isomorphism. These disadvantages make the concept of dg-schemes too rigid. A summary of our analysis here can be found in [66, Section 1.2].

Remark 2.5.10. There is another definition for derived schemes in [64, Definition 2.1]. A derived scheme X there consists of a scheme $\pi_0 X$ and a (pre)sheaf $\mathcal{O}_X$ on the site of affine open subschemes of $\pi_0 X$ with values in $\mathbf{cdgA}^{\leq 0}_k$ such that $H^0(\mathcal{O}_X) = \mathcal{O}_{\pi_0 X}$ and all $H^n(\mathcal{O}_X)$ are quasi-coherent $\mathcal{O}_{\pi_0 X}$-modules for $n \leq 0$. [50, Theorem 6.42] shows this definition is weakly equivalent to Definition 2.5.8 here. In this way the abstract concept of derived schemes become more vivid and the comparison between derived schemes and classical schemes is more obvious.

A derived scheme above is affine if the underlying scheme $\pi_0 X$ is an affine scheme. With this definition [64, p186] shows the category of affine derived schemes is equivalent to $\mathbf{dAff}_k$ at the level of infinite categories. But unfortunately in [64] Toën does not describe the infinite category structure on the category of derived schemes since it's complicated and totally in the context of [50] using hypergroupoids. If we describe morphisms between derived schemes directly, we may make the same mistake in the case of dg-schemes since the category of derived schemes does not admit a model structure and its homotopy theory is difficult to describe under this definition.

In Proposition 2.4.6 we have said the theory of commutative algebras is contained in the homotopy theory of cdgas. [68, Section 2.2.4] shows how the theory of higher stacks can be contained in the theory of derived stacks. Moreover we can associate every derived stack with a higher stack as its truncation. Then a derived stack is a derived scheme if and only if its truncation is a classical scheme.

As for the relationship between dg-schemes and derived schemes, you can consult with [62, Example 2 in Sec. 4.3].

Remark 2.5.11 (Truncation)**.** The natural embedding $j : \mathbf{Aff}_k \to \mathbf{dAff}_k$ induces a functor

$$j^* : sPr(\mathbf{dAff}_k) \to sPr(\mathbf{Aff}_k), \;\; \mathcal{F} \mapsto \mathcal{F} \circ j$$

On the other hand for any simplicial presheaf $\mathcal{G} \in sPr(\mathbf{Aff}_k)$,

$$\begin{array}{ccc} \mathbf{Aff}_k^{op} & \xrightarrow{\;\mathcal{G}\;} & s\mathbf{Set} \\ {\scriptstyle j}\downarrow & \Downarrow & \nearrow_{j_!\mathcal{G}} \\ (\mathbf{dAff_k})^{op} & & \end{array}$$

$j_!\mathcal{G}$ is defined to be the left Kan extension (Definition A.3.1) of $\mathcal{G}$ along j which exits since $s\mathbf{Set}$ is cocomplete. Then we obtain an adjoint pair

$$j_! : sPr(\mathbf{Aff}_k) \rightleftarrows sPr(\mathbf{dAff}_k) : j^*$$

This adjoint pair is a Quillen pair. Moreover by properties of left Bousfield localization [67, Section 4.8] and [68, Section 2.2.4] show this pair can be defined on local model categories. Therefore we obtain an adjunction between usual higher stacks and derived stacks.

$$i = \mathbb{L}j_! : \mathbf{St}_k \rightleftarrows \mathbf{dSt}_k : t_0 = \mathrm{Ho}(j^*)$$

[68, Lemma 2.2.4.1] shows the functor $i = \mathbb{L}j_!$ is fully faithful. Hence the theory of higher stacks can be contained in the theory of derived stacks. [68, Lemma 2.2.4.2] shows j^* is left and right Quillen, and preserves weak equivalences. That's why we can just define t_0 as $\mathrm{Ho}(j^*)$. More properties of this adjunction can be found in [68, Section 2.2.4]. For example they all preserve (derived) n-geometric stacks.

3 Derived Critical Loci

We always suppose k **is a field of characteristic** 0. Our main reference in this section is [70].

3.1 Koszul Resolution

Definition 3.1.1. Let R be a commutative k-algebra and M be an R-module. The *tensor algebra* of M over R is defined to be the non-commutative R-algebra $T^\bullet(M) := \oplus_{n\geq 0}T^n(M)$ where $T^n(M) = M^{\otimes n}$ and $T^0(M) = R$. The multiplication in $T(M)$ is that

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m \in T^{n+m}(M)$$

The *exterior algebra* $\bigwedge^\bullet(M) = T(M)/\sim$ where the two sided ideal is generated by $x \otimes x \in T^2(M)$. Elements in $\bigwedge^n(M) = T^n(M)/\sim$ are denoted by $x_1 \wedge \cdots \wedge x_n$.

The *symmetric algebra* $\mathrm{Sym}^\bullet(M) = T(M)/\sim$ where the two sided ideal is generated by $x \otimes y - y \otimes x$. The image $x_1 \otimes \cdots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted by $x_1 \cdots x_n$. $\mathrm{Sym}^\bullet(M)$ is commutative.

Fact 3.1.2. Since $(x+y)\otimes(x+y) = x\otimes x + x \otimes y + y \otimes x + y \otimes y = 0$ in $\bigwedge^\bullet(M)$, $x \wedge y = -y \wedge x$ in $\bigwedge^\bullet(M)$. Therefore the multiplication in $\bigwedge^\bullet(M)$ is graded commutative.

Example 3.1.3. If M is a free R-module of finite rank i.e. $M = Rx_1 \oplus \cdots \oplus Rx_n$. Then $\bigwedge^k(M)$ is an R-module freely generated by elements $x_{i_1} \wedge \cdots \wedge x_{i_k}$ where $i_1 < \cdots < i_k$. And $\mathrm{Sym}^\bullet(M) = R[x_1, \cdots, x_n]$.

Recall a *differential graded algebra* (dga) is a (cochain) complex $A^\bullet$ with the unit $1 \in A^0$ and multiplication $\cdot : A^n \times A^m \to A^{n+m}$ satisfying $d(ab) = (da)b + (-1)^{|a|}a(db)$ where $|a|$ is the degree of a. Also we may write $A^\bullet = \oplus_{n\in\mathbb{Z}}A^n$. Another equivalent definition says a dga is actually a *differential graded category* with only one object. A dga is commutative if it's graded commutative i.e. $ab = (-1)^{|a||b|}ba$.

Definition 3.1.4. Let R be a commutative k-algebra and E be an R-module with an R-linear map $x : E \to R$. Then the *Koszul complex* $K(E, x)$ is defined to be the following complex

$$K(E,x) := \cdots \longrightarrow \bigwedge^k E \xrightarrow{d_k} \cdots \xrightarrow{d_2} \bigwedge^1 E = E \xrightarrow{x} R \longrightarrow 0$$

where

$$d_k : \bigwedge^k E \to \bigwedge^{k-1} E,\ \ e_1 \wedge \cdots \wedge e_k \mapsto \sum_{i=1}^k (-1)^{i+1}x(e_i)e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k$$

Clearly $d_{k-1}d_k = 0$ and $d(e \wedge e') = de \wedge e' + (-1)^{|e|}e \wedge e'$, Moreover since the wedge product is graded commutative, the Koszul complex $K(R, E; x)$ is actually a commutative differential graded algebra (cdga).

Remark 3.1.5. In the following we suppose R is a commutative k-algebra and P is a finite projective R-module which is also equivalent to saying the module sheaf $\widetilde{P}$ is a locally free sheaf of finite rank over $\mathrm{Spec}R$ [3, Tag00NV]. Let $P^\vee := \mathrm{Hom}_R(P, R)$ and $S = \mathrm{Sym}^\bullet_R(P^\vee)$. For $n \leq 0$, we define $(S \otimes \bigwedge^\bullet P^\vee)^n = S \otimes_R \bigwedge^{-n} P^\vee$. Then the *fancy Koszul (cochain) complex* $K(R; P)$ of S-modules is defined to be following (cochain) complex.

$$K(R;P) := \cdots \longrightarrow S \otimes_R \bigwedge^n P^\vee \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-2}} S \otimes_R \bigwedge^1 P^\vee \xrightarrow{d^{-1}} S \longrightarrow 0$$

where

$$d : x_1 \cdots x_p \otimes (y_1 \wedge \cdots \wedge y_n) \mapsto \sum_i (-1)^{i+1} x_1 \cdots x_p y_i \otimes (y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n)$$

From Remark 2.4.2 we see the fancy Koszul cochain complex $K(R; P)$ is actually the free graded commutative algebra $\mathcal{S}(P^\vee \xrightarrow{\mathrm{id}} P^\vee)$ where the cochain complex is centered at $[-1, 0]$. Then $K(R; P)$ is cofibrant in

$\mathbf{cdgA}_R^{\leq 0}$ and $\mathbf{cdgA}_S^{\leq 0}$, since $\mathcal{S}$ is left Quillen. Note that $S \to K(R;P)$ is a cofibration in $\mathbf{cdgA}_R^{\leq 0}$ because $S = \mathcal{S}(P^\vee)$ and the following map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P^\vee \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \cdots & \longrightarrow & 0 & \longrightarrow & P^\vee & \xrightarrow{\mathrm{id}} & P^\vee \end{array}$$

in $\mathbf{Ch}_R^{\leq 0}$ is a cofibration.

Theorem 3.1.6. *The Koszul (cochain) complex $K(R;P)$ is the projective resolution of R when viewing R as an S-module via the natural projection $S \to R$.*

Proof. Since P is locally free, then $P^\vee$ will be locally free as well. Therefore $\bigwedge^n P^\vee$ is locally free over $\mathrm{Spec}R$ and $S \otimes_R \bigwedge^n P^\vee$ is finite projective over S. Next we prove the following (cochain) complex

$$\cdots \longrightarrow S \otimes_R \bigwedge^n P^\vee \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-2}} S \otimes_R \bigwedge^1 P^\vee \xrightarrow{d^{-1}} S \xrightarrow{\text{proj}} R \longrightarrow 0$$

has trivial (co-)homology groups.

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & S \otimes_R \bigwedge^n P^\vee & \xrightarrow{d} & \cdots & \longrightarrow & S \otimes_R \bigwedge^1 P^\vee & \longrightarrow & S & \xrightarrow{\text{proj}} & R & \longrightarrow & 0 \\ & \swarrow_{s_n} & & & & \swarrow_{s_1} & & \swarrow_{s_0} & & \swarrow_{s_{-1}} & & & \\ \cdots & \longrightarrow & S \otimes_R \bigwedge^n P^\vee & \longrightarrow & \cdots & \longrightarrow & S \otimes_R \bigwedge^1 P^\vee & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

$s_{-1} : R \to S$ is the natural embedding and

$$s_n : S \otimes_R \bigwedge^n P^\vee \to S \otimes_R \bigwedge^{n+1} P^\vee, \;\; x_1 \cdots x_p \otimes (y_1 \wedge \cdots \wedge y_n) \mapsto \sum_{i=1}^{p} x_1 \cdots \hat{x}_i \cdots x_p \otimes (x_i \wedge y_1 \wedge \cdots \wedge y_n)$$

Then

$$\begin{aligned} x_1 \cdots x_p \otimes (y_1 \wedge \cdots \wedge y_n) &\xrightarrow{d} x_1 \cdots x_p y_i \otimes \sum_i (-1)^{i+1} (y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n) \\ &\xrightarrow{s_{n-1}} \sum_i (-1)^{1+i} x_1 \cdots x_p \otimes (y_i \wedge y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n) \\ &\quad + \sum_i (-1)^{i+1} \sum_j x_1 \cdots \hat{x}_j \cdots x_p y_i \otimes (x_j \wedge y_1 \wedge \hat{y}_i \wedge \cdots \wedge y_n) \end{aligned}$$

and

$$\begin{aligned} x_1 \cdots x_p \otimes (y_1 \wedge \cdots \wedge y_n) &\xrightarrow{s_n} \sum_j x_1 \cdots \hat{x}_j \cdots x_p \otimes (x_j \wedge y_1 \wedge \cdots \wedge y_n) \\ &\xrightarrow{d} \sum_j x_1 \cdots x_j \cdots x_p \otimes (y_1 \wedge \cdots \wedge y_n) \\ &\quad + \sum_j \sum_i (-1)^i x_1 \cdots \hat{x}_j \cdots x_p y_i \otimes (x_j \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n) \end{aligned}$$

This means $s_{n_1} \circ d + d \circ s_n = (p+n)\mathrm{id}$ on $\mathrm{Sym}_R^p(P^\vee) \otimes_R (\bigwedge^n P^\vee)$. For any $u \in S \otimes_R \bigwedge^n P^\vee$, $u = \sum_p u_p$ where $u_p \in \mathrm{Sym}_R^p(P^\vee) \otimes_R \bigwedge^n P^\vee$. Then $du_i \in \mathrm{Sym}_R^{p+1}(P^\vee) \otimes_R \bigwedge^{n-1} P^\vee$. Hence $du = 0 \Leftrightarrow du_p = 0$ for all p. And moreover $(s_{n-1}d + ds_n)u_p = (p+n)u_p$. Since $\mathrm{char}\, k = 0$, $p+n$ is a unit. Then for any u satisfying $du = 0$, $\mathrm{cls}\, u_p = \mathrm{cls}\, ds_n(\frac{u_p}{p+n}) = 0$ where cls means the image is in the homology groups. Therefore $\mathrm{cls} u = 0$.

This proves $K(R;P)$ is the projective resolution of R of S-modules. □

Corollary 3.1.7. *The map $K(R;P) \to R$ in $\mathbf{cdgA}_S^{\leq 0}$ is actually a cofibrant replacement.*

3.1.1 Derived Zero Loci

Given an element $s \in P$, then there will exist an R-linear map $s : P^\vee \to R$ sending any $f \in P^\vee$ to $f(s) \in R$. And we obtain the Kozul complex

$$K(R,P;s) = K(P^\vee, s) = \cdots \longrightarrow \bigwedge^n P^\vee \xrightarrow{d} \cdots \longrightarrow \bigwedge^1 P^\vee = P^\vee \xrightarrow{s} R \longrightarrow 0$$

We can view it as a cochain complex whose module at the $-n$ degree for $n \geq 0$ is $\bigwedge^n P^\vee$.

Since $S = \mathrm{Sym}^\bullet_R(P^\vee)$, there exists a map $\varphi_s : S \to R$ sending $x_1 \cdots x_p \in S$ to $x_1(s)\cdots x_p(s)$ where $x_i : P \to R \in P^\vee$. Via this map, R will be an S-algebra and we denote it as R_s to distinguish it from the original S-algebra structure on R obtained from the natural projection. Viewing R_s as a cdga over S centered at the degree 0, we have the tensor product $R_s \otimes_S K(R;P)$ on the category of S-cdgas where

$$(R_s \otimes K(R;P))^{-n} = R_s \otimes_S S \otimes_R (\bigwedge^n P^\vee) \cong R_s \otimes_R (\bigwedge^n P^\vee) \cong \bigwedge^n P^\vee$$

And we obtain the canonical isomorphism $R_s \otimes_S K(R;P) \cong K(R,P;s)$ between S-cdgas where $x_1 \cdots x_p \in S$ acts on $\bigwedge^n P^\vee$ by sending $y_1 \wedge \cdots \wedge y_n$ to $x_1(s)\cdots x_p(s) y_1 \wedge \cdots \wedge y_n$.

Remark 3.1.8. The above statements imply we can compute the homotopy pushout $R_s \xleftarrow{\varphi_s} S \xrightarrow{\mathrm{proj}} R$ in a way that according to Theorem A.4.10 it's just the pushout of $R_s \xleftarrow{\varphi_s} S \longrightarrow K(R;P)$ since $\mathbf{cdgA}_R^{\leq 0}$ is left proper. Moreover we have

$$R_s \otimes_S^{\mathbb{L}} R = R_s \otimes_S K(R;P) = K(R,P;s)$$

Dually in the category $\mathbf{dAff}_R$ of affine derived scheme, the homotopy fiber product

$$\begin{array}{ccc} & & \mathrm{Spec}R \\ & & \big\downarrow{\scriptstyle 0} \\ \mathrm{Spec}R_s & \xrightarrow[s]{} & \mathrm{Spec}S \end{array}$$

is just $Z^h(s) = \mathrm{Spec}K(R.P;s)$ where s and 0 correspond with φ_s and the natural projection $S \to R$ respectively. $Z^h(s)$ is called the *derived zero locus* of the section s.

In [70] Vezzosi computes this homotopy pullback in the category $\mathbf{dSt}_k$ of derived stacks, in which the result is $\mathbb{R}\underline{\mathrm{Spec}}K(R.P;s)$. To obtain this result we need to prove $\mathbb{R}\underline{\mathrm{Spec}}$ commutes with homotopy pullbacks. As we have known in Remark 2.5.6,

$$\underline{\mathrm{Spec}} = \underline{h} : \mathbf{dAff}_k \to \mathbf{dAff}_k^\wedge, \;\; \mathrm{Spec}A \mapsto \mathrm{Hom}_{\mathbf{dAff}_k}(\Gamma(-), \mathrm{Spec}A)$$

and then $\mathbb{R}\underline{\mathrm{Spec}} = \underline{\mathrm{Spec}}R$ where R is the fibrant replacement functor. In Section 2.2 we have proved the functor $\underline{h}$ preserves fibrant objects and trivial fibrations. Moreover if the codoamin is $sPr(\mathbf{dAff}_k)_{proj}$ not the Bousfield localization $\mathbf{dAff}_k^\wedge$, $\underline{h}$ will preserve fibrations. This is also true when passing to $\mathbf{dAff}_k^\wedge$ if we only deal with maps between fibrant objects, since Theorem A.6.17 tells us in the Bousfield localization fibrations between fibrant objects are precisely fibrations in the original category. Therefore $\mathbb{R}\underline{\mathrm{Spec}}$ preserves weak equivalences by Lemma A.3.2 (Ken Brown) and fibrations. Moreover the hom functor $\mathrm{Hom}_{\mathbf{dAff}_k}(\Gamma(X), -)$ preserves arbitrary limits especially fiber products. All these prove $\mathbb{R}\underline{\mathrm{Spec}}$ commutes with homotopy pullbacks.

Remark 3.1.9. In classical algebraic geometry on a scheme X, any locally free $\mathcal{O}_X$-module sheaf $\mathcal{E}$ of finite rank can be associated with a (geometric) vector bundle $f : \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee \to X$[16] in the sense that there

[16] In many references the actual (geometric) vector bundle is defined as $\mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}$ since it can be easily generalized to arbitrary quasi-coherent module sheaves and induce an anti-equivalence between quasi-coherent module sheaves and abstract vector bundles [3, Tag01M1]. But in the case of finite locally free sheaf $\mathcal{E}$, we have $\mathcal{E}^{\vee\vee} = \mathcal{E}$. Therefore our definition here will just induce an equivalence not an anti-equivalence.

exists an open covering $\{U_i | i \in I\}$ for X such that $f^{-1}(U_i)$ in $\mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee$ is isomorphic to some affine space $\mathbb{A}^n$.

Here for the affine case $\mathrm{Spec}S = \mathrm{Spec}\mathrm{Sym}_R P^\vee$ is a vector bundle on $\mathrm{Spec}R$ and $s \in \widetilde{P}(\mathrm{Spec}R)$ is a global section for the canonical structure of $\mathrm{Spec}S$ over $\mathrm{Spec}R$. Therefore $Z^h(s)$ gets the name the derived zero locus of the section s.

The definition for derived zero locus can also be generalized to the case of schemes. Suppose X is a scheme over k and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module sheaf whose corresponding (geometric) vector bundle is $E = \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee$. Let $s \in \mathcal{E}(X)$ and this will define a map $s : X \to E$ which is glued by $s|\mathrm{Spec}R$. We can compute the homotopy fiber product $X_s \xrightarrow{s} E \xleftarrow{0} X$ in the category of derived schemes or derived stacks $\mathbf{dSt}_k$. In the sense of Remark 2.5.10, $K(X;\mathcal{E})$ will be a derived scheme with the underlying topological space X and the homotopy fiber product

$$X_s \times_E^h X = X_s \times_E K(X;\mathcal{E}) = K(X,\mathcal{E};s) = (\bigwedge^{\bullet} \mathcal{E}^\vee, d_s)$$

is a derived scheme with the underlying topological space $X_s \times_E X$ the usual fiber product.

We need to explain the fiber product of derived schemes in more details. $K(X;\mathcal{E})$ is a sheaf of cdgas on X which locally is $K(R;P)$. To see it's a derived scheme we need to compute the cohomology group $H^0 K(X;\mathcal{E})$. But for the affine case we see

$$H^0 K(R;P) = \mathrm{Sym}_R^{\bullet} P^\vee / \mathrm{Sym}_R^1 P^\vee = R$$

Therefore the underlying topological space for $K(X;\mathcal{E})$ is glued by $\mathrm{Spec}R$ which is just X. Then $(K(X;\mathcal{E}), X)$ is a derived scheme.

As for $K(X,\mathcal{E};s)$, it's actually also a sheaf of cdgas on the underlying topological space X. Now we want to compute its cohomology group $H^0 K(X,\mathcal{E};s)$. Again we deal with this problem locally. Clearly $H^0 K(R.P;s) = R/\mathrm{im}\ s$ where $s : P^\vee \to R$ sends any $f \in P^\vee$ to $f(s)$. The usual fiber product is that $\mathrm{Spec}R_s \otimes_S R$. View R as S/I where $I = \mathrm{Sym}_R^1 P^\vee$ then the tensor product should be R_s/IR_s which is just $R/\mathrm{im}\ s$. $\mathrm{Spec}R/\mathrm{im}\ s$ is a closed subscheme of $\mathrm{Spec}R$. Gluing affine cases we obtain a closed subscheme $X_s \times_E X$ of X. Then $K(X,\mathcal{E};s)$ is naturally a sheaf on $X_s \times_E X$.

Our discussion here in this section can also be translated into the area *derived differential geometry* where the algebra model is not cdgas but *differential graded C^∞-rings*. Details of this viewpoint can be found in the nLab page of seminar notes for derived critical loci written by Urs Schreiber.

3.2 Derived Critical Loci

Suppose X is a smooth scheme over k. Then $\Omega^1_{X/k}$ the *sheaf of differential forms of degree* 1 is finite locally free. Note that for any scheme the diagonal map $\Delta : X \to X \times_k X$ is a locally closed immersion. Let $W \subseteq X \times_k X$ be an open subscheme such that $\Delta : X \to W$ is a closed immersion and its corresponding ideal is $\mathcal{I}$. Then $\Omega^1_{X/k} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$. So $\Omega^1_{X/k}$ is glued by the module of Kähler differentials of rings. In some textbook $\Omega^1_{X/k}$ is also called the *cotangent sheaf* of X and its corresponding bundle $\mathrm{T}^*X := \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/k})^\vee$ is the *cotangent bundle* over X. Similarly the dual sheaf $\mathrm{T}_X := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ is called the *tangent sheaf* and its corresponding bundle $\mathrm{T}_*X := \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \Omega^1_{X/k}$ is the *tangent bundle*.

Given a global section $f \in \mathcal{O}_X(X)$ which is actually equivalent to a map $f : X \to \mathbb{A}^1_k$, $df \in \Omega^1_{X/k}(X)$ is the Kähler differential of f.

Definition 3.2.1. The *derived critical locus* $\mathrm{Crit}^h(f)$ of f is defined to be the derived zero locus $Z^h(df)$ of df. Then its corresponding algebra will be $K(X, \Omega^1_{X/k}; df) = (\bigwedge^{\bullet} \mathrm{T}_X, df)$.

3.2.1 Gerstenhaber Algebra

Definition 3.2.2. For an object A in $\mathbf{cdgA}_k^{\leq 0}$, a *Lie bracket of degree* 1 on A is a bilinear map $[\cdot,\cdot] : A\otimes A \to A[1]$ satisfying

- (graded antisymmetry): $[a,b] = -(-1)^{(|a|+1)(|b|+1)}[b,a]$
- (Jacobi relation): $[a,[b,c]] = [[a,b],c] + (-1)^{(|a|+1)(|b|+1)}[b,[a,c]]$
- (differential): $d[a,b] = [da,b] + (-1)^{|a|+1}[a,db]$

This Lie bracket make A be a *differential graded Lie algebra (dgLa) of degree* 1. Moreover we say such A is a *differential graded Gerstenhaber algebra (dgGa)* if the bracket is a biderivation of the product i.e.

$$[a, b\cdot c] = [a,b]\cdot c + (-1)^{|b|(|a|+1)} b\cdot [a,c]$$

Remark 3.2.3. For axioms of a dgLa of degree 1, with the axiom of graded antisymmetry the second axiom of Jacobi relation is equivalent to the following Jacobi equation of degree 1

$$(-1)^{(|a|+1)(|c|+1)}[a,[b,c]] + (-1)^{(|c|+1)(|b|+1)}[c,[a,b]] + (-1)^{(|b|+1)(|a|+1)}[b,[c,a]] = 0$$

And for a dgGa, we also have

$$[a\cdot b, c] = a\cdot [b,c] + (-1)^{|b|(|c|+1)}[a,c]\cdot b$$

Example 3.2.4 (Schouten bracket)**.** [17] Suppose $X = \mathrm{Spec} R$ is a smooth scheme over k and $\mathrm{T}_R := (\Omega^1_{R/k})^\vee$ the tangent module. Then we have a Koszul algebra $K(\mathrm{T}_R) = \bigoplus_{n\leq 0} \bigwedge^{-n} \mathrm{T}_R$ where we ignore the differential and only consider the level of graded algebras. Since

$$\mathrm{T}_R = \mathrm{Hom}_R(\Omega^1_{R/k}, R) = \mathrm{Der}_k(R,R)$$

for any element $f\in \mathrm{T}_R$ we can also regard it as a derivation $R\to R$. So for $f\in\mathrm{T}_R$ and $a\in R$ we define $[f,a] = f(a)\in R$. Note that here $f(a)$ is actually $f(da)$ where $d: R\to \Omega^1_{R/k}$ is the canonical derivation. For two derivations $f,\ g\in \mathrm{T}_R$ we let $[f,g] = f\circ g - g\circ f$. Note that it's not difficult to check $[f,g]$ is actually a k-derivation. In general we define

$$[f_1\wedge\cdots\wedge f_n, a] = \sum_{i=1}^n (-1)^{n-i} f_i(a) f_1\wedge\cdots\wedge \hat{f_i}\wedge\cdots\wedge f_n$$

for $f_i\in\mathrm{T}_R$, $a\in R$ and

$$[f_1\wedge\cdots\wedge f_n, g_1\wedge\cdots\wedge g_m] = \sum_{i=1}^n\sum_{j=1}^m (-1)^{i+j}[f_i,g_j]\wedge f_1\wedge\cdots\wedge\hat{f_i}\wedge\cdots\wedge f_n\wedge g_1\wedge\cdots\wedge\hat{g}_j\wedge\cdots\wedge g_m$$

for $f_i,\ g_j\in\mathrm{T}_R$. These definitions will make $K(\mathrm{T}_R)$ be a Gerstenhaber algebra with zero differentials (check!). For Gerstenhaber condition it's enough to notice

$$\begin{aligned}
&[f_1\wedge\cdots\wedge f_n, g_1\wedge\cdots\wedge g_m\wedge h_1\wedge\cdots\wedge h_l]\\
=&\sum_{i=1}^n\sum_{j=1}^m (-1)^{i+j}[f_i,g_j]\wedge f_1\wedge\cdots\wedge\hat{f_i}\wedge\cdots\wedge f_n\wedge g_1\wedge\cdots\wedge\hat{g}_j\wedge\cdots\wedge g_m\wedge h_1\wedge\cdots\wedge h_l\\
&+\sum_{i=1}^n\sum_{k=1}^l (-1)^{i+m+k}[f_1,h_k]\wedge f_1\wedge\cdots\wedge\hat{f_i}\wedge\cdots\wedge f_n\wedge g_1\wedge\cdots\wedge g_m\wedge h_1\wedge\cdots\wedge\hat{h}_k\wedge\cdots\wedge h_l
\end{aligned}$$

[17] [43, Example VII.15]

and

$$[f_1 \wedge \cdots \wedge f_n, ab] = \sum_{i=1}^{n} (-1)^{n-1} (af_i(b) + bf_i(a)) f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n$$

The two cases above are then clear. Next we only need to consider $[f, ag]$ for $f,\ g \in \mathrm{T}_R$ and $a \in R$ which is just

$$[f, ag] = f(ag) - (ag)f = f(a)g + a(fg) - a(gf) = [f, a]g + a[f, g]$$

This can be generalized for $[f_1 \wedge \cdots \wedge f_n, ag_1 \wedge \cdots \wedge g_m]$.

We omit the process of checking them satisfying the Jacobi relation.

Theorem 3.2.5. *Let $X = \mathrm{Spec}R$ be a smooth affine scheme over k and $\alpha \in \Omega^1_{R/k}$. Then with the Schouten bracket described above, the Koszul cdga $K(R, \Omega^1_{R/k}; \alpha)$ is a dgGa.*

Proof. For $f_i \in \mathrm{T}_R$ and $a \in R$ we have

$$\begin{aligned} d[f_1 \wedge \cdots \wedge f_n, a] &= d\big(\sum_{i=1}^{n} (-1)^{n-i} f_i(a) f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n\big) \\ &= \sum_{i=1}^{n} (-1)^{n-i} f_i(a) \Big(\sum_{j=1}^{i-1} (-1)^{j+1} \alpha(f_j) f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n \\ &\quad + \sum_{j=i+1}^{n} (-1)^{j} \alpha(f_j) f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_n \Big) \end{aligned}$$

and on the other hand we have

$$\begin{aligned} [d(f_1 \wedge \cdots \wedge f_n), a] &= [\sum_{j=1}^{n} (-1)^{j+1} \alpha(f_j) f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_n, a] \\ &= \sum_{j=1}^{n} (-1)^{j+1} \alpha(f_j) \Big(\sum_{i=1}^{j-1} (-1)^{n-1-i} f_i(a) f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_n \\ &\quad + \sum_{i=j+1}^{n} (-1)^{n-i} f_i(a) f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n \Big) \end{aligned}$$

Therefore we have an equation

$$d[f_1 \wedge \cdots \wedge f_n, a] = [d(f_1 \wedge \cdots \wedge f_n), a]$$

For $f,\ g \in \mathrm{T}_R$, $\alpha[f, g] = \alpha(fg - gf) = f(dg(\alpha)) - g(df(\alpha))$ where $d : R \to \Omega^1_{R/k}$ is the canonical derivation. On the other hand

$$[\alpha(f), g] + [f, \alpha(g)] = -[g, \alpha(f)] + [f, \alpha(g)] = -g(df(\alpha)) + f(dg(\alpha))$$

More general result

$$d[f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_m] = [d(f_1 \wedge \cdots \wedge f_n), g_1 \wedge \cdots \wedge g_m] + (-1)^{n+1} [f_1 \wedge \cdots \wedge f_n, d(g_1 \wedge \cdots \wedge g_m)]$$

can be checked. The left side is actually

$$
\begin{aligned}
& d[f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_m] \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{i+j}\Big(\alpha([f_i,g_j]) \wedge f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_m \\
& \qquad + \sum_{k=1}^{i-1}(-1)^{k+2}\alpha(f_k)[f_i,g_j] \wedge f_1 \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots g_m \\
& \qquad + \sum_{k=i+1}^{n}(-1)^{k+1}\alpha(f_k)[f_i,g_j] \wedge f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge \hat{g}_j \wedge \cdots g_m \\
& \qquad + \sum_{k=1}^{j-1}(-1)^{n+k+1}\alpha(g_k)[f_i,g_j] \wedge f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{g}_k \wedge \cdots \wedge \hat{g}_j \wedge \cdots g_m \\
& \qquad + \sum_{k=j+1}^{m}(-1)^{n+k}\alpha(g_k)[f_i,g_j] \wedge f_1 \wedge \cdots \hat{f}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge \hat{g}_k \wedge \cdots g_m\Big)
\end{aligned}
$$

For the right side

$$
\begin{aligned}
& [d(f_1 \wedge \cdots \wedge f_n), g_1 \wedge \cdots \wedge g_m] \\
=& [\sum_{k=1}^{n}(-1)^{k+1}\alpha(f_k) f_1 \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_m] \\
=& \sum_{k=1}^{n}(-1)^{k+1}\alpha(f_k)\Big(\sum_{i=1}^{k-1}\sum_{j=1}^{m}(-1)^{i+j} f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_m \\
& \qquad + \sum_{i=k+1}^{n}\sum_{j=1}^{m}(-1)^{i-1+j} f_1 \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_m\Big) \\
& + (-1)^{(n-1)(m+1)}\sum_{k=1}^{n}(-1)^{k+1}[\alpha(f_k), g_1 \wedge \cdots \wedge g_m] \wedge f_1 \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge f_n
\end{aligned}
$$

$(-1)^{n+1}[f_1 \wedge \cdots \wedge f_n, d(g_1 \wedge \cdots \wedge g_m)]$ can be computed similarly. We omit this computation. ☐

BV Formalization

Definition 3.2.6. A *Batalin-Vilkovisky algebra* or *BV-algebra* simply is a Gerstenhaber algebra with a linear map $\Delta : A \to A[1]$ of degree 1 such that

- $\Delta \circ \Delta = 0$;
- it satisfies $\Delta(ab) = \Delta(a)b + (-1)^{|a|}a\Delta(b) + (-1)^{|a|}[a,b]$.

Remark 3.2.7. For a BV-algebra we have

$$\Delta(\Delta(a)b) + (-1)^{|a|}\Delta(a\Delta(b)) + (-1)^{|a|}\Delta([a,b]) = 0$$

which is just

$$\Delta([a,b]) = [\Delta(a),b] + (-1)^{|a|+1}[a,\Delta(b)]$$

Then Δ becomes a derivation of degree 1 for the bracket.

Definition 3.2.8. A BV-algebra is a *differential graded BV-algebra (dgBV-algebra)* or a *Beilinson-Drinfeld (BD) algebra* if it satisfies $d\Delta + \Delta d = 0$.

An example for BV-algebra appears in differential geometry or more precisely mathematical physics.

Definition 3.2.9. For a smooth manifold M of dimension n, a *top form* is differential n-form in $\Omega^n(M)$ which is called a *volume form* if it's nowhere vanishing. In this case M is *orientable*.

Example 3.2.10. A *symplectic manifold* consists of a pair (M, ω) where M is a smooth manifold of dimension $2n$ and ω is a closed non-degenerate 2-form which is called a *symplectic form*. Every symplectic manifold admits a volume form $\frac{\omega^n}{n!}$ which is called *symplectic volume* of (M, ω).

Example 3.2.11. Suppose M is a smooth manifold of dimension n which admits a volume form. Our discussion for derived critical loci being a Gerstenhaber algebra is also valid here. With Schouten bracket the cdga $\bigwedge^{\bullet}\Gamma(\mathrm{T}_X)$ for multivector fields is a Gerstenhaber algebra. For any volume form μ, its contraction induces isomorphisms

$$\mu : \Omega^{n-i}(M) \xrightarrow{\sim} \bigwedge^{i} \Gamma(\mathrm{T}_X)$$

for $0 \leq i \leq n$. In this case we define $\Delta = \mu \circ d_{\mathrm{dR}} \circ \mu^{-1}$ and then $\bigwedge^{\bullet}\Gamma(\mathrm{T}_X)$ will be a BV-algebra. This appears in BV-formalization.

As for Calabi-Yau folds, a nowhere vanishing holomorphic volume exists and therefore the statement above is also true in this case.

In mathematical physics there is a concept of BV-BRST formalization to deal with the critical loci of an action functional S and the symmetries of S where the latter is often a Lie groupoid or more generally a L_∞-groupoid. Then the obtained BV-BRST complex consists of two parts. For $n \leq 0$ the BV part is just what we discussed above and for positive degrees the BRST part is the associated Chevalley-Eilenberg algebra of the L_∞-algebroid which is the (homotopy) quotient of the gauge group. Details of these can be found in the nLab page of BV-BRST formalization.

3.3 Loop Spaces

In algebraic topology there is a notion of loop spaces. For a topological space X, its *loop space* $\Omega_x X$ consists of pointed maps $(S^1, 0) \to (X, x)$ with the compact-open topology. More generally a *free loop space* for X is defined as $\mathcal{L}X := \mathrm{Map}(S^1, X)$ with the compact-open topology. Note that an elementary theorem in algebraic topology says S^1 is the first Eilenberg-Maclane space $K(\mathbb{Z}, 1)$. This motivates us to define the *derived loop stack* for a stack $\mathcal{F}$ as the derived mapping space from $K(\mathbb{Z}, 1)$ to $\mathcal{F}$. In this section we will describe this structure clearly and then talk about the derived zero locus of the section 0.

In $s\mathbf{Set}$, $N(\mathbb{Z})$ the nerve of $\mathbb{Z}$ represents a model for $K(\mathbb{Z}, 1)$. We use the symbol $B\mathbb{Z}$ or simply S^1 to denote the constant derived stack of $K(\mathbb{Z}, 1)$ which means it's the stackification of the constant simplicial presheaf sending any object to the simplicial set $K(\mathbb{Z}, 1)$.

Remark 3.3.1. This definition above having a historical flavor can play an important role in classifying cohomology groups. For example we suppose the Grothendieck site $\mathcal{C}$ is just the category $\mathbf{Top}$ of topological spaces with the usual open immersions as the pretopology. Suppose $K(\mathbb{Z}, n)$ is the Eilenberg-Maclane simplicial set such that its homotopy groups are all trivial except $\pi_n K(\mathbb{Z}, n) = \mathbb{Z}$. Actually $K(\mathbb{Z}, n)$ can be easily obtained via Dold-Kan correspondence. Then we consider the constant simplicial presheaf $\mathcal{C} \to s\mathbf{Set}$ sending any topological space to $K(\mathbb{Z}, n)$. We also use the symbol $K(\mathbb{Z}, n)$ to denote the stackification of the constant simplicial presheaf in the local model category $\mathbf{St}(\mathcal{C}) = \mathrm{Ho}(sPr(\mathcal{C})^{loc}_{proj})$. Then for any topological space X viewing it as a representable simplicial presheaf which is actually a usual sheaf, we have

$$\pi_i \mathbb{R}\mathrm{Map}(X, K(\mathbb{Z}, n)) \cong H^{n-i}(X, \mathbb{Z})$$

where the right side is computed as the sheaf cohomology.

Remark 3.3.2 (Mapping stacks)**.** In the usual topos theory, a Grothendick topos i.e. $Shv(\mathcal{C})$ for some Grothendieck site $\mathcal{C}$ is Cartesian closed which means for any two sheaves F and G there is an internal hom sheaf $\mathbf{Hom}(F, G)$ such that for any other sheaf H we have a natural isomorphism

$$\mathrm{Hom}_{Shv(\mathcal{C})}(H \times F, G) \cong \mathrm{Hom}_{Shv(\mathcal{C})}(H, \mathbf{Hom}(F, G))$$

Details can be found in [41, Section III.6]. This is also true for higher topoi. In our cases here this means the category of stacks $\mathbf{St}_k$ and derived stacks $\mathbf{dSt}_k$ are Cartesian closed. Let us use $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ the symbol of local model category to denote $\mathbf{St}_k$ or $\mathbf{dSt}_k$. Then the statement above means for any two objects $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ the following functor

$$\mathrm{Ho}(sPr(\mathcal{C})_{loc}) \to \mathrm{Ho}(s\mathbf{Set}), \;\; \mathcal{H} \mapsto \mathbb{R}\mathrm{Map}(\mathcal{H} \times^h \mathcal{F}, \mathcal{G})$$

is representable by some object $\mathbf{Map}(\mathcal{F}, \mathcal{G})$ in $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$, which means

$$\mathbb{R}\mathrm{Map}(\mathcal{H} \times^h \mathcal{F}, \mathcal{G}) \cong \mathbb{R}\mathrm{Map}(\mathcal{H}, \mathbf{Map}(\mathcal{F}, \mathcal{G}))$$

Note that in the following we always use the symbol $\mathbf{Map}(\mathcal{F}, \mathcal{G})$ (resp. $\mathbb{R}\mathbf{Map}(\mathcal{F}, \mathcal{G})$) to mean the *mapping stack* (resp. *derived mapping stack*) in $\mathbf{St}_k$ (resp. $\mathbf{dSt}_k$), so that we can distinguish them.

Here we deal with the homotopy theory of (derived) stacks directly and there is naturally a question that whether this Cartesian closed property can be lifted to the local model category $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ of simplicial presheaves in other words whether $\mathrm{Ho}(sPr(\mathcal{C})_{loc})$ is a symmetric monoidal model category with the usual product. For $sPr(\mathcal{C})^{loc}_{inj}$ the answer is positive but it's not true for $sPr(\mathcal{C})^{loc}_{proj}$. However, the two model categories are Quillen equivalent and therefore passing to homotopy theories i.e. the level of ∞-categories, the right derived internal hom functor for $sPr(\mathcal{C})^{loc}_{inj}$ gives the correct answer for mapping stacks. More precisely in $sPr(\mathcal{C})^{loc}_{inj}$ there is an internal hom functor

$$\mathbf{Hom} : sPr(\mathcal{C})^{loc}_{inj} \times sPr(\mathcal{C})^{loc}_{inj} \to sPr(\mathcal{C})^{loc}_{inj}, \quad \mathbf{Hom}(\mathcal{F}, \mathcal{G})(c) = \mathrm{Map}(c \times \mathcal{F}, \mathcal{G})$$

such that with the usual direct product $sPr(\mathcal{C})^{loc}_{inj}$ is a symmetric monoidal model category in the sense of Definition A.5.8. And moreover the mapping stack functor $\mathbf{Map}$ is computed as $\mathbb{R}\mathbf{Hom}$. Note that

$$\mathbb{R}\mathbf{Hom}(\mathcal{F}, \mathcal{G}) = \mathbf{Hom}(\mathcal{F}, R_{inj}\mathcal{G})$$

where $R_{inj}\mathcal{G}$ is a fibrant replacement for $\mathcal{G}$ in $sPr(\mathcal{C})^{loc}_{inj}$. Details of these can be found in [67, Section 3.6].

There is a more general discussion in [38] where Lurie shows for any *presentable ∞-category* when an ∞-functor from it to the ∞-category of simplicial sets is representable ([38, Proposition 5.5.2.2]). More internal properties of ∞-topoi can be found in [67] and [38].

If $\mathcal{F}$ and $\mathcal{G}$ are stacks in $\mathbf{St}_k$, then their derived mapping stack is defined to be $\mathbb{R}\mathbf{Map}(i(\mathcal{F}), i(\mathcal{F}))$ where $i : \mathbf{St}_k \to \mathbf{dSt}_k$ is an embedding in Remark 2.5.11. Moreover the truncation of derived mapping stacks will be isomorphic to the usual one, i.e.

$$t_0\mathbb{R}\mathbf{Map}(i(\mathcal{F}), i(\mathcal{G})) \cong \mathbf{Map}(\mathcal{F}, \mathcal{G})$$

Definition 3.3.3. Let $\mathcal{F}$ be a stack in $\mathbf{St}_k$. The *loop stack* for $\mathcal{F}$ is defined to be $\mathcal{LF} := \mathbf{Map}(S^1, \mathcal{F})$ and its *derived loop stack* is $\mathbb{R}\mathcal{LF} := \mathbb{R}\mathbf{Map}(S^1, i(\mathcal{F}))$. If $\mathcal{F}$ is a derived stack in $\mathbf{dSt}_k$, then the *derived loop stack* of $\mathcal{F}$ is defined as $\mathbb{R}\mathcal{LF} := \mathbb{R}\mathbf{Map}(S^1, \mathcal{F})$.

Although here we define the concept of (derived) loop spaces in a really general context, we are more interested in elementary cases like (derived) schemes.

Remark 3.3.4. To compute the derived loop stack $\mathbb{R}\mathcal{LF}$ for a derived stack $\mathcal{F}$, we need to figure out the circle object S^1 clearly. Notice that in $s\mathbf{Set}$, $\Delta^1/\partial\Delta^1$ serves a model for $K(\mathbb{Z}, 1)$. Moreover generally $\Delta^1/\partial\Delta^1$ is just the homotopy colimit $* \coprod^{\mathbb{L}}_{* \coprod^{\mathbb{L}} *} *$. First note that in $s\mathbf{Set}$ every object is cofibrant, then $* \coprod^{\mathbb{L}} * = * \coprod *$. Next to compute the following homotopy colimit

$$\begin{array}{ccc} * \coprod * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

since $s\mathbf{Set}$ is left proper, we replace $* \coprod * \to *$ by a cofibration $* \coprod * \hookrightarrow \Delta^1$ where it's obvious to see Δ^1 is weakly equivalent to $*$. Therefore the homotopy colimit is actually

$$\Delta^1 \coprod_{* \coprod *} * = \Delta^1 / \partial \Delta^1$$

which is just the circle object in $\mathrm{Ho}(s\mathbf{Set})$. So in an ∞-topos $\mathcal{C}$ its *homotopical circle* is actually defined as

$$S^1 := * \coprod_{* \coprod *} *$$

where colimits are computed in the context of ∞-categories. Then for any object X in this ∞-topos, its loop space will be defined as

$$\mathcal{L}X := \mathbf{Map}(S^1, X) = X \times_{X \times X} X$$

In our case here, the ∞-topos is $\mathbf{St}_k$ and $\mathbf{dSt}_k$. Then the derived loop stack for a derived stack $\mathcal{F}$ is actually

$$\mathbb{R}\mathcal{L}\mathcal{F} = \mathcal{F} \times^h_{\mathcal{F} \times^h \mathcal{F}} \mathcal{F}$$

Remark 3.3.5. As shown in the nLab page of "Slice Action", for any object X in an ∞-topos $\mathcal{C}$ its free loop space $\mathcal{L}X$ is naturally a group object in $\mathcal{C}/X$ whose group structure comes from the composition and inversion of loops. In the classical case, we let $\mathcal{C}$ be $\mathbf{Top}$ and the loop space $\mathbf{Map}(S^1, X)$ is regraded as an object in $\mathbf{Top}/X$ via an evaluation map of a fixed point of S^1. Then this gives a group structure on $\mathbf{Map}(S^1, X)$ clearly which is similar to $\Omega_x X$. There is a similar discussion for derived zero loci in [70], where $0^2 = \mathbb{R}\underline{\mathrm{Spec}}K(X, \mathcal{E}; 0)$ the self intersection of the zero section has the structure of Segal monoid in $\mathbf{dSt}_k/X$.

Example 3.3.6. Suppose A is a cdga in $\mathbf{cdgA}_k^{\leq 0}$ and $X = \mathrm{Spec}A$ is its dual object in $\mathbf{dAff}_k$. Then we see viewing as a derived stack, the derived loop space for X is

$$\mathbb{R}\mathcal{L}X = \mathbb{R}\underline{\mathrm{Spec}}(A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}} A} A)$$

since we have proved $\mathbb{R}\underline{\mathrm{Spec}}$ commutes with homotopy fiber products. This construction will compute the Hochschild (co)homology for A [36, Proposition 1.1.13].

If $A = k[x]$ the affine line, then its derived loop space is just its self intersection as discussed in Example 2.5.7.

Next we suppose A is a regular commutative algebra which means the corresponding affine scheme is smooth over k, then the *Hochschild-Kostant-Rosenberg (HKR) isomorphism* $HH_n(A) \cong \Omega^n_{A/k}$ (see [36, Section 3.4]) computes the derived loop space as the de Rham dg-algebra

$$\mathbb{R}\underline{\mathrm{Spec}}(\cdots \xrightarrow{0} \bigwedge^n \Omega^1_{A/k} \xrightarrow{0} \cdots \xrightarrow{0} \Omega^1_{A/k} \xrightarrow{0} A) = \mathbb{R}\underline{\mathrm{Spec}}\mathcal{S}(\Omega^1_{A/k}[1]) = \mathbb{R}\underline{\mathrm{Spec}}K(A, (\Omega^1_{A/k})^\vee; 0)$$

There is a generalization in [7] and [69], which deal with more general cases using the technique of *affinization*. It's shown in [7, Proposition 3.1] (originally in Toën's paper *Champs affines*, arXiv:math/0012219) that there is an adjunction

$$\mathbb{L}\mathcal{O} : \mathbf{dSt}_k \rightleftarrows \mathbf{dAff}_k : \mathbb{R}\underline{\mathrm{Spec}}$$

Then for any derived stack $\mathcal{F}$ its affinization is defined to be $\mathbb{R}\underline{\mathrm{Spec}}\mathbb{L}\mathcal{O}(\mathcal{F})$. [7, Proposition 1.1] shows for a derived scheme X we have an equivalence

$$HC(\mathbb{L}\mathcal{O}(X)) \simeq \mathcal{S}_{\mathbb{L}\mathcal{O}(X)}(\Omega_X[1])$$

where the left side is the Hochschild chain and $\Omega_X[1]$ is the cotangent complex for X.

3.4 Shifted Symplectic Structures

In this section we only talk about *shifted symplectic structures* in the affine case i.e. for cdgas. More general statements for derived Artin stacks can be found in [48].

As talked before for a cdga A, its differential forms $\Omega^1_{A/k}$ of degree 1 is a differential graded A-module and the derivation $d_{\mathrm{dR}} : A \to \Omega^1_{A/k}$ is actually a chain map. In this case the de Rham cocomplex

$$\Omega^\bullet_A := A \xrightarrow{d_{\mathrm{dR}}} \Omega^1_{A/k} \xrightarrow{d_{\mathrm{dR}}} \Omega^2_{A/k} \to \cdots$$

where $\Omega^n_{A/k} = \bigwedge^n_A \Omega^1_{A/k}$ is actually the following bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{-2} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{1,-2}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{2,-2}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \cdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
A^{-1} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{1,-1}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{2,-1}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \cdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
A^{0} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{1,0}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \Omega^{2,0}_{A/k} & \xrightarrow{d_{\mathrm{dR}}} & \cdots
\end{array}$$

Here we also write $\Omega^0_{A/k}$ for the cdga A. Note that there is a sign trick. We know d_{dR} is a map of chain complexes but in the definition of a bicomplex it requires $d^v d^h + d^h d^v = 0$ so that the induced *total complexes* will really be complexes. Therefore in the column $\Omega^{p,\bullet}_{A/k}$ the differential in it is defined as $(-1)^p d$. Then its total (co)complex $\mathrm{Tot}^{\prod}\Omega^\bullet_A$ is defined such that

$$(\mathrm{Tot}^{\prod}\Omega^\bullet_A)^n := \prod_{p+q=n} \Omega^{p,q}_{A/k}$$

and the differential on $\Omega^{p,q}_{A/k}$ is actually $d_{\mathrm{dR}} + (-1)^p d$.

For a complex or a bicomplex, there is a filtration defined as follows

$$F^p(\cdots \to V^n \to V^{n+1} \to \cdots) = \cdots \to 0 \to V^p \to V^{p+1} \to \cdots$$

Here for algebraic de Rham complex, such filtration is called *Hodge filtration*. Classically for a smooth algebraic variety X over k especially over $\mathbb{C}$, the sheaf $Z^p\Omega^\bullet_X$ will be quasi-isomorphic to $(F^p\Omega^\bullet_X)[p]$ in the sense of *hypercohomology*, which means the sheaf cohomology of $Z^p\Omega^\bullet_X$ will be isomorphic to the hypercohomology of algebraic Hodge filtration $(F^p\Omega^\bullet_X)[p]$.

Remark 3.4.1. For our purpose here that the construction should be in the homotopy sense, we replace a cdga A by its cofibrant replacement QA and consider $\Omega^\bullet_{QA}$ or the cotangent complex $\mathbb{L}_A = \Omega^1_{QA/k} \otimes_{QA} A$ and its wedge products $\bigwedge^\bullet \mathbb{L}_A$. The two points are equivalent and we explain why they are equivalent. Given a map $f : A \to B$ in $\mathbf{cdgA}_k^{\leq 0}$, we have a Quillen adjunction

$$f^* := - \otimes_A B : \mathbf{dgMod}_A^{\leq 0} \rightleftarrows \mathbf{dgMod}_B^{\leq 0} : f_*$$

and moreover if f is a quasi-isomorphism, then this will induce an equivalence on the homotopy level. Since f_* creates weak equivalence i.e. a morphism $g \in \mathbf{dgMod}_B^{\leq 0}$ is a weak equivalence if and only if $f_*(g)$ is a weak equivalence, we only need to prove for any cofibrant object M in $\mathbf{dgMod}_A^{\leq 0}$ the adjunction unit $M \to M \otimes_A B$ is a weak equivalence.[18] By the functorial cofibrant replacement construction in the small object argument, we may just suppose M is *quasi-free* i.e. of the form $\oplus_{i\in I} A[n_i]$ for some integers n_i. Then we have an isomorphism

$$\mathrm{Tor}_0^{H^*A}(H^*(M), H^*(B)) \xrightarrow{\sim} H^*(M \otimes_A B)$$

and moreover since $H^*A \cong H^*B$, the left side above is actually $H^*(M)$. Therefore this proves when QA is a cofibrant replacement of A in $\mathbf{cdgA}_k^{\leq 0}$ and $f : QA \to A$ is a weak equivalence, we have a Quillen equivalence (f^*, f_*), from which we can identify $\Omega^1_{QA/k}$ with $\mathbb{L}_A$.

Definition 3.4.2 (closed p-forms)**.** For a cdga A or an affine derived scheme $\mathrm{Spec}A$, the *complex of closed p-forms* is defined to be $\mathcal{A}^{p,cl}(A) := (\mathrm{Tot}^{\prod} F^p\Omega^\bullet_{QA})[p]$.

The shifted version for constructions talked above for a cdga A or an affine derived scheme $\mathrm{Spec}A$ is defined as follows

- the *complex of n-shifted p-forms* is $\mathcal{A}^p(A, n) := \Omega^p_{QA}[n]$
- the *complex of n-shifted closed p-forms* is $\mathcal{A}^{p,cl}(A, n) := (\mathrm{Tot}^{\prod} F^p\Omega^\bullet_{QA})[p + n]$

An n-shifted closed p-form ω on $\mathrm{Spec}A$ is an element in $Z^0\mathcal{A}^{p,cl}(A, n)$ and more precisely $\omega = (\omega_i)_{i\geq 0}$ where $\omega_i \in \Omega^{p+i,n-i}_{QA}$ satisfies $d_{\mathrm{dR}}\omega_i + (-1)^{p+i+1}d\omega_{i+1} = 0$ and $d\omega_0 = 0$.

Classically a *symplectic structure* ω on a smooth scheme X over k is a closed algebraic 2-form in $\Omega^2_{X/k}(X)$ such that the induced map $\theta_\omega : \mathrm{T}_X \to \Omega^1_{X/k}$ is a sheaf isomorphism. We know for a k-algebra R, $\Omega^n_R = \bigwedge^n \Omega^1_{R/k}$ consists of all alternating functions $\prod_{i=1}^n \mathrm{T}_R \to R$. Then for Ω^2_R any element in it will define an alternating map $\mathrm{T}_R \times \mathrm{T}_R \to R$ and especially it induces $\mathrm{T}_R \to \Omega^1_{R/k} = \mathrm{T}_R^\vee$. This explains how θ_ω works. We can generalize this definition to the derived case.

Definition 3.4.3. For a cdga A, its *tangent complex* is defined to be

$$\mathbb{T}_A = \mathbb{L}_A^\vee = \mathbb{R}\mathrm{Hom}_A(\mathbb{L}_A, A)$$

where $\mathbb{R}\mathrm{Hom}$ is the right derived hom functor in $\mathbf{dgMod}_A$. An *n-shifted symplectic structure* on A is an n-shifted closed two-form $\omega \in \mathcal{A}^{2,cl}(A, 2)$ on A such that the underlying two-form ω_0 induces an equivalence

$$\theta_\omega : \mathbb{T}_A \xrightarrow{\sim} \mathbb{L}_A[n]$$

We finish this section by the following theorem.

Theorem 3.4.4. [19] *For a function $f : X \to \mathbb{A}^1_k$ where X is a smooth scheme, the derived critical locus $\mathrm{Crit}^h(f)$ is -1-shifted symplectic.*

[18] It's Proposition 2.3 in the nLab page of Quillen equivalence.
[19] [70, Example 4.8]

The strategy here is to show cotangent bundles are 0-shifted symplectic and the *derived Lagrangian intersection* over an n-shifted symplectic variety is $(n-1)$-shifted symplectic [48, Theorem 2.10].

Remark 3.4.5. In this remark we talk about symplectic structures on cotangent bundles.

Considering the case of differential geometry first, for a smooth manifold M of dimension n it cotangent bundle T^*M is of dimension $2n$. The projection map $\pi : \mathrm{T}^*M \to M$ sends (x, ξ) where $\xi \in T_x^*M$ is a covector at x. We know differential coordinates $(d\xi_i)_{1\le i\le n}$ form a basis on the cotangent space and ξ can be expressed as $\sum_{i=1}^n \xi_i d\xi_i$ uniquely. The *canonical* 1*-form* (or *tautological* 1*-form*, or *Liouville-Poincaré* 1*-form*) is defined to be $\beta = \sum_{i=1}^n x_i d\xi_i \in \Omega^1(\mathrm{T}^*M)$ where we also use $d\xi_i$ to mean the differential of the coordinate function at $d\xi_i$ on T*M. Its differential $d\beta = \sum_{i=1}^n dx_i \wedge d\xi_i$ gives a symplectic form on T^*M.

Then for a smooth scheme X we can reduce it to the affine case. So we can just suppose $X = \mathrm{Spec}R$ is a smooth affine scheme over k and the *cotangent algebra* is $\mathrm{T}^*R := \mathrm{Sym}_R(\Omega^1_{R/k})^\vee$. Since $\Omega^1_{R/k}$ is locally free, reduced to some basic open subset of $\mathrm{Spec}R$ we may just assume $\Omega^1_{R/k}$ is free. In this case $(\Omega^1_{R/k})^\vee$ can be canonically identified with $\Omega^1_{R/k}$. Now let $\{\xi_i = dx_i\}$ be a basis for $\Omega^1_{R/k}$ over R where $x_i \in R$. Then $d\xi_i$ is an element in $\Omega^1_{\mathrm{T}^*R/k}$. Note that there is a natural embedding $R \to \mathrm{T}^*R$. So we can view x_i's as elements in T^*R. This gives a *tautological* 1*-form* $\sum_i x_i d\xi_i$ in $\Omega^1_{\mathrm{T}^*R/k}$ whose differential $\omega_0 = \sum_i dx_i \wedge d\xi_i$ gives a symplectic form on T^*R.

Definition 3.4.6. [20] Assume X is a smooth scheme over k with a symplectic structure ω_0. A subscheme $f : L \hookrightarrow X$ admits a *Lagrangian structure* if it's a closed immersion and $f^*\omega_0 = 0$.

Then for a 1-form $\alpha \in \Omega^1_{R/k}$, the induced two closed immersions $0 : \mathrm{Spec}R \hookrightarrow \mathrm{Spec}\mathrm{T}^*R$ and $\alpha : \mathrm{Spec}R_\alpha \hookrightarrow \mathrm{Spec}\mathrm{T}^*R$ are Lagrangian. The former is clear. To see the latter we just need to notice that $\alpha^*(d\xi_i) = \alpha_i dx_i$. For the Lagrangian embedding $f : L \to X$ there is an equivalence $\mathbb{T}_L \xrightarrow{\sim} \mathbb{L}_f[-1]$ [11, Example 2.13] where $\mathbb{L}_f$ is the relative cotanegnt complex.

Proof of Theorem 3.4.4. Let α be a 1-form in $\Omega^1_{R/k}$ and we prove the derived zero locus $Z^h(\alpha)$ is -1-shifted symplectic. Generally for $Z^h(\alpha) = \mathrm{Spec}K(R, \Omega^1_{R/k}; \alpha)$, $\omega_0 = \sum_i d_{\mathrm{dR}} x_i \wedge d_{\mathrm{dR}} \xi_i$ gives a -1-shifted symplectic structure. Here symbols come from Remark 3.4.5.

We have the following homotopy pullback diagram

$$\begin{array}{ccc} Z^h(\alpha) & \xrightarrow{\ i\ } & \mathrm{Spec}R \\ {\scriptstyle j}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 0} \\ \mathrm{Spec}R_\alpha & \xrightarrow[\alpha]{} & \mathrm{Spec}\mathrm{T}^*R \end{array}$$

which is also a homotopy pushout diagram in $\mathbf{cdgA}_k^{\le 0}$. Since the functor of cotangent complexes is left Quillen, it preserves homotopy pushouts. Therefore if we suppose $q = \alpha \circ j \simeq 0 \circ i$, we have the following homotopy pushout

$$\begin{array}{ccc} q^*\mathbb{L}_{\mathrm{T}^*R} & \longrightarrow & i^*\mathbb{L}_R \\ \big\downarrow & \ulcorner & \big\downarrow \\ j^*\mathbb{L}_{R_\alpha} & \longrightarrow & \mathbb{L}_{Z^h(\alpha)} \end{array}$$

which is actually the homotopy coequalizer

$$q^*\mathbb{L}_{\mathrm{T}^*R} \underset{0}{\overset{(j_*, i_*)}{\rightrightarrows}} j^*\mathbb{L}_{R_\alpha} \oplus i^*\mathbb{L}_R \longrightarrow \mathbb{L}_{Z^h(\alpha)}$$

From this point we see

$$q^*\mathbb{L}_{\mathrm{T}^*R} \longrightarrow j^*\mathbb{L}_{R_\alpha} \oplus i^*\mathbb{L}_R \longrightarrow \mathbb{L}_{Z^h(\alpha)}$$

[20] [11, Example 2.13]

is a cofiber sequence (the composition is null-homotopic) hence a distinguished triangle in the derived category $D(Z^h(\alpha))$. Applying the internal hom functor, we obtain a new distinguished triangle of tangent complexes

$$\mathbb{T}_{Z^h(\alpha)} \longrightarrow j^*\mathbb{T}_{R_\alpha} \oplus i^*\mathbb{T}_R \longrightarrow q^*\mathbb{T}_{\mathrm{T}^*R}$$

For the composition map $Z^h(\alpha) \xrightarrow{q} \mathrm{Spec}\mathrm{T}^*R \to \mathrm{Spec}k$ it induces a distinguished triangle [53, Theorem 5.1]

$$q^*\mathbb{L}_{\mathrm{T}^*R} \longrightarrow \mathbb{L}_{Z^h(\alpha)} \longrightarrow \mathbb{L}_{Z^h(\alpha)/\mathrm{T}^*R}$$

But according to [53, Theorem 5.3], there is an equivalence

$$j^*\mathbb{L}_{R_\alpha/\mathrm{T}^*R} \oplus i^*\mathbb{L}_{R/\mathrm{T}^*R} \simeq \mathbb{L}_{Z^h(\alpha)/\mathrm{T}^*R}$$

So that we have the following distinguished triangle

$$\mathbb{L}_{Z^h(\alpha)}[-1] \longrightarrow j^*\mathbb{L}_{R_\alpha/\mathrm{T}^*R}[-1] \oplus i^*\mathbb{L}_{R/\mathrm{T}^*R}[-1] \longrightarrow q^*\mathbb{L}_{\mathrm{T}^*R}$$

We know $\mathrm{Spec}R$ and $\mathrm{Spec}R_\alpha$ are Lagrangian subschemes and this will induce the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{T}_{Z^h(\alpha)} & \longrightarrow & j^*\mathbb{T}_{R_\alpha} \oplus i^*\mathbb{T}_R & \longrightarrow & q^*\mathbb{T}_{\mathrm{T}^*R} \\
\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathbb{L}_{Z^h(\alpha)}[-1] & \longrightarrow & j^*\mathbb{L}_{R_\alpha/\mathrm{T}^*R}[-1] \oplus i^*\mathbb{L}_{R/\mathrm{T}^*R}[-1] & \longrightarrow & q^*\mathbb{L}_{\mathrm{T}^*R}
\end{array}$$

The induced map $\mathbb{T}_{Z^h(\alpha)} \to \mathbb{L}_{Z^h(\alpha)}[-1]$ is therefore an equivalence by the property of triangulated categories. This proves $Z^h(\alpha)$ is -1-shifted symplectic. □

Remark 3.4.7. There are more general statements in [48] where we can define the critical locus for any smooth geometric stack. For a derived Artin stack there is a concept of *n-shifted cotangent (derived) stack* ([48, Definition 1.20]) which admits an n-shifted tautological 1-form and therefore it's n-shifted symplectic. A general definition for *Lagrangian structures* is given in [48, Definition 2.8] and then the critical locus is a Lagrangian intersection based on a 0-shifted symplectic cotangent stack hence having a -1-shifted symplectic structure ([48, Corollary 2.11]).

A Homotopical Algebra

In this appendix we give a detailed introduction to the abstract theory of homotopical algebra necessary for this thesis and references here contain [28], [31] and [55].

Roughly speaking, in a model category there are three important classes of morphisms, called fibrations, cofibrations and weak equivalences respectively. They reveal the lifting properties and quasi-isomorphisms in a given category. In fact the most important class of morphisms above is that of weak equivalences since a category may admit different model structures with equivalent homotopy theory. In general fibrations and cofibrations are defined to help us study properties of weak equivalences, which is just similar to that when studying manifolds coordinates are not a must but they can really help us study manifolds and do some computations. For a given manifold, there are many choices of local coordinates and for a class of weak equivalences there may also exist some choices of fibrations and cofibrations to make them form a model category, which means in general for a category the model structure is not unique. But for all these model structure, their homotopy theories are equivalent since it's defined to be the localization with respect to weak equivalences. Therefore we can say homotopical algebra is to study weak equivalences and some properties invariant under weak equivalences. Hence it's natural to look at localization categories first.

Definition A.0.1. Let $\mathcal{C}$ be a category with small Hom sets, and $\mathcal{W}$ be a set of morphisms. Then there will exist *the category of fractions* (or called *localization category*) $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ with respect to $\mathcal{W}$ and a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that:

(1) For any $f \in \mathcal{W}$, $\gamma(f)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.

(2) For any functor $G : \mathcal{C} \to \mathcal{D}$ such that $\forall f \in \mathcal{W}, G(f)$ are isomorphims, there is a unique functor $F : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $F \circ \gamma = G$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \mathcal{C}[\mathcal{W}^{-1}] \\ {\scriptstyle G}\downarrow & \swarrow {\scriptstyle \exists! F} & \\ \mathcal{D} & & \end{array}$$

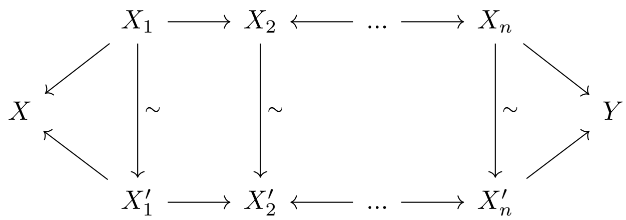
Remark A.0.2. Obviously we know adding isomorphisms to $\mathcal{W}$ will not affect the universal category $\mathcal{C}[\mathcal{W}^{-1}]$. And we can enlarge $\mathcal{W}$ to become a subcategory of $\mathcal{C}$ and this will not affect the localization category as well. Moreover such localization category always exists though there will be some set theoretic difficulties. We give the proof of existence here.

Proof. We construct the category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ as follows. We consider a directed graph $\mathcal{G}$ first. The vertexes in $\mathcal{G}$ are just objects in $\mathcal{C}$. For any map $f : A \to B$ in $\mathcal{W}$, we add a map $f^{-1} : B \to A$. The set of oriented edges in $\mathcal{G}$ consists of edges in $\mathcal{C}$ and those f^{-1}. Then, we identify the path $f \circ f^{-1}$ with id_B , $f^{-1} \circ f$ with id_A , $g \circ h$ with gh where $g, h \in \mathrm{Mor}(\mathcal{C})$, $\mathrm{id} \circ g$ with g, $h \circ \mathrm{id}$ with h where $g, h \in \mathcal{G}$ and $g \circ (h \circ l)$ with $(g \circ h) \circ l$ for any oriented edges $g, h, l \in \mathcal{G}$. The quotient directed graph will be the category $\mathcal{C}[\mathcal{W}^{-1}]$.

There is another description of the Hom set of localization categories. In $\mathcal{C}[\mathcal{W}^{-1}]$ every morphism $X \to Y$ has the following form:

$$X \longleftarrow X_1 \longrightarrow X_2 \longleftarrow X_3 \longrightarrow ... \longleftarrow X_n \longrightarrow Y$$

where left arrows are in $\mathcal{W}$, right arrows in $\mathrm{Mor}(\mathcal{C})$. Sequences obtained by adding identities are viewed the same as the original one. Hence for any two sequences we can add identities to them to make them having the same number of objects. So that we can define a complicated equivalence relation among such sequences. This equivalence relation is generated by the following diagram



where vertical morphisms are in $\mathcal{W}$. Two sequences are relevant if there is such a commutative diagram above in $\mathcal{C}$. The equivalence is generated by these relations. □

Remark A.0.3. The Definition A.0.1 is strict since all such $\mathcal{C}[\mathcal{W}^{-1}]$'s are isomorphic which means there will exist one-to-one relations on objects and morphisms. But for categories we only consider equivalence classes of them not isomorphism classes. Therefore we give a weaker definition here.

Let $\mathrm{Hom}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ be the full subcategory of the category of functors between $\mathcal{C}$ and $\mathcal{D}$ consisting of those functors taking every morphism in $\mathcal{W}$ to isomorphisms. Then $\gamma : \mathcal{C} \to \mathbf{C}[\mathcal{W}^{-1}]$ is defined to have the universal property

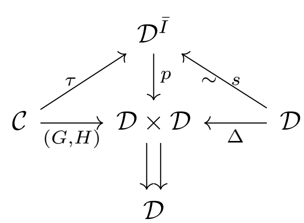
$$\gamma^* : \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$$

where γ^* is defined by composition and it's an equivalence between categories not an isomorphism. This will define $\mathcal{C}[\mathcal{W}^{-1}]$ up to equivalence.

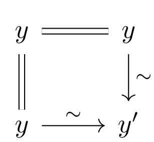
Remark A.0.4. There is also a simplicial way to deal with the localization such that we obtain a simplicial category $L\mathcal{C}$ satisfying $\pi_0 L\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$. Its name is called the *hommock localization*. Details can be found in [18].

Example A.0.5. The category of all small categories is denoted by $\mathbf{Cat}$ and The set of equivalences between categories is denoted by $\mathcal{W}$. For any two functor $F, G : \mathcal{C} \to \mathcal{D}$ we say they are equivalent if there is a natural isomorphism $\tau : F \overset{\sim}{\Rightarrow} G$. It's obvious to see it's actually an equivalence relation among $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ and it's preserved by compositions. Hence we can define the homotopy category $\mathrm{Ho}(\mathbf{Cat})$ to have the same objects as $\mathbf{Cat}$ and its morphism sets are the equivalence classes described above. Then $\mathrm{Ho}(\mathbf{Cat}) \cong \mathbf{Cat}[\mathcal{W}^{-1}]$.

Proof. The category generated by one isomorphism $1 \xrightarrow{\sim} 2$ is denoted by $\bar{I}$. We write the functor category of $\mathcal{D}$ over $\bar{I}$ as $\mathcal{D}^{\bar{I}}$. It's enough to prove for any functor $\mathcal{F} : \mathbf{Cat} \to \mathcal{E}$ taking categorical equivalences to isomorphisms, if there is a natural isomorphism between functors $\tau : G \overset{\sim}{\Rightarrow} H$ where $G, H \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$, then $\mathcal{F}(G) = \mathcal{F}(H)$.



where $\tau : a \mapsto (\tau_a : G(a) \xrightarrow{\sim} H(a))$, $p : (x \xrightarrow{\sim} y) \mapsto (x, y)$, $\Delta(x) = (x, x)$ and $s : x \to (\mathrm{id}_x : x \xrightarrow{\sim} x)$. Note s is actually a categorical equivalence. Obviously it's fully faithful. For any $y \xrightarrow{\sim} y'$ in $\mathcal{D}^{\bar{I}}$



We see s is also essentially surjective hence a categorical equivalence. Then $\mathcal{F}(s)$ is an isomorphism. We are done. $\square$

Remark A.0.6. Note that the localization of categories often cause set theoretic problems, which means the morphism set of $\mathcal{C}[\mathcal{W}^{-1}]$ may be a proper class. Example A.0.5 is a special case since the morphism set of $\mathrm{Ho}(\mathbf{Cat})$ is small. Actually $\mathbf{Cat}$ is a model category and the localization of model categories with respect to weak equivalences will not cause this set theoretic problem. What's more apart from the technique of model categories, there is a useful technique to solve this set theoretic problem as well which is actually earlier than Quillen's work and whose motivation coming from the localization of non-commutative rings is more natural. Such technique is called *calculus of fractions*. Details can be found in [21, Chapter 1] or any text book about *derived categories*.

Next to talk about model categories, let us begin with the *factorization system* first.

A.1 Factorization Systems

In a category $\mathcal{C}$, let $i : A \to B$ and $p : X \to Y$ be two morphisms in it. We say i has the *left lifting property* with respect to (LLP wrt) p or p has the *right lifting property* with respect to (RLP wrt) i if for any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there exists $h : B \to X$ making the new diagram commutative. If $\mathcal{F}$ is a class of morphisms in $\mathcal{C}$, we use $l(\mathcal{F})$ (resp. $r(\mathcal{F})$) to denotes the class of morphisms having the LLP (resp. RLP) wrt all morphisms in $\mathcal{F}$. [21] If we have two classes $\mathcal{F}$, $\mathcal{G}$ of morphisms such that every map in $\mathcal{F}$ has the LLP wrt all maps in $\mathcal{G}$, then we write $\mathcal{F} \boxslash \mathcal{G}$. This symbol comes from [55].

Definition A.1.1. In a category $\mathcal{C}$, we say $f : X \to X'$ is a *retract* of $g : Y \to Y'$ if there is the following comutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

such that $r \circ i = \mathrm{id}_X, r' \circ i' = \mathrm{id}_{X'}$.

Lemma A.1.2. *In a category $\mathcal{C}$, if $f : X \to Y$ can be factored as $f = p \circ i$ where f has the RLP (resp. LLP) wrt i (resp. p), then f is a retract of p (resp. i).*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p} & \\ & Z & \end{array}$$

[21] In different textbooks, notations of $l(\mathcal{F})$ and $r(\mathcal{F})$ are different (see [14], [31]] and [55]). Here we follow [14] since we think his notations are the simplest.

Proof. We only need to assume $f \in r(i)$, since on the other hand we can deal with this problem in $\mathcal{C}^{op}$.

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[p]{} & Y \end{array}$$

This diagram above implies

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown} & & \\ X & \xrightarrow{i} & Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ Y & = & Y & = & Y \end{array}$$

□

Now we want to talk about properties of $l(\mathcal{F})$.

Definition A.1.3. A class of morphisms $\mathcal{F}$ is closed under pushouts if given any pushout diagram,

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

that $f \in \mathcal{F}$ implies $f' \in \mathcal{F}$; it's closed under retraction if in the diagram of Definition A.1.1, that $g \in \mathcal{F}$ implies $f \in \mathcal{F}$; it's closed under coproducts if given $f_i : X_i \to Y_i$ belonging to $\mathcal{F}$ for $i \in I$, then so does

$$\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

$\mathcal{F}$ is closed under *transfinite compositions* if for every well-ordered set I with the initial element 0, for any functor $X : I \to \mathcal{C}$ such that for any element $i \in I, i \neq 0$, the colimit $\operatorname{colim}_{j<i} X(j)$ exists and the induced map

$$\operatorname*{colim}_{j<i} X(j) \to X(i)$$

is in $\mathcal{F}$, then the colimit $\operatorname{colim}_{i\in I} X(i)$ exists and the morphism $X(0) \to \operatorname{colim}_{i\in I} X(i)$ belongs to $\mathcal{F}$.

The class of morphisms satisfying properties above is called *saturated*.

Remark A.1.4. Actually given a class of morphisms $\mathcal{F}$, if it's closed under pushouts and transfinite compositions, then it will also be closed under coproducts.

Proof. Suppose there are morphisms $f_i : X_i \to Y_i$ belonging to $\mathcal{F}$ for $i \in I$. From the well-ordering axiom, we may assume I is well-ordered and 0 is its initial element. Firstly, we have a pushout diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & \coprod_{i\in I} X_i \\ {\scriptstyle f_0}\downarrow & & \downarrow \\ Y_0 & \longrightarrow & Z_0 \end{array}$$

where Z_0 is actually $Y_0 \coprod_{i\in I\setminus 0} X_i$. If 1 is the successor of 0, then we have the pushout

$$\begin{array}{ccc} X_1 & \longrightarrow & Z_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow \\ Y_1 & \longrightarrow & Z_1 \end{array}$$

where $X_1 \to Z_0$ is just $X_1 \to \coprod_{i\in I} X_i \to Z_0$. For a limit number i, we let Z_i be the following poushout

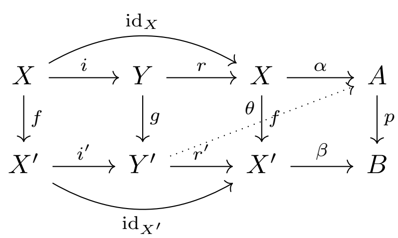
$$\begin{array}{ccc} X_i & \longrightarrow & \operatorname{colim}_{j<i} Z_j \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow \\ Y_i & \longrightarrow & Z_i \end{array}$$

Since $\mathcal{F}$ is closed under pushouts, $\coprod_{i\in I} X_i \to Z_0$, $Z_0 \to Z_1$ and $\operatorname{colim}_{j<i} Z_j \to Z_i$ all belong to $\mathcal{F}$. Note that $\operatorname{colim}_{j<i} Z_j$ is actually $(\coprod_{j<i} Y_j) \coprod (\coprod_{i'\geq i} X_{i'})$. Finally we see $\operatorname{colim}_{i\in I} Z_i = \coprod_{i\in I} Y_i$, which can also be proved by the universal property of coproducts of Y_i's. Then that $\mathcal{F}$ is closed under transfinite compositions implies $\coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is in $\mathcal{F}$. ☐

Theorem A.1.5. *In a category $\mathcal{C}$, for any class of morphisms $\mathcal{F}$, $l(\mathcal{F})$ is saturated.*

Proof. From Remark A.1.4 we only to check $l(\mathcal{F})$ is closed under retraction, pushouts and transfinite compositions.

Step 1 (retraction). If f is the retraction of g where $g \in l\mathcal{F}$, given any $p : A \to B$ belonging to $\mathcal{F}$

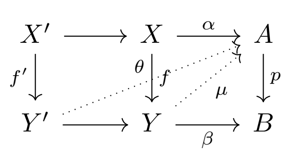


Then there will be a lifting $\theta : Y' \to A$ and $\theta \circ i'$ will give the solution of the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle p} \\ X' & \xrightarrow{\beta} & B \end{array}$$

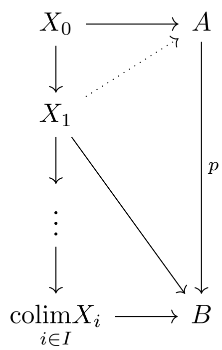
Hence $f \in l(\mathcal{F})$.

Step 2 (pushouts). Look at the pushout diagram in Definition A.1.3 where $f \in l(\mathcal{F})$ and f' is the pushout of f. Also given the lifting problem above



θ is induced by f. Using the universal property of pushouts, we see the solution μ exists.

Step 3 (transfinite compositions). If 1 is the successor of 0,



the lifting respect to $X_0 \to X_1$ exists. Especially for any $X_i \to X_{i+1}$ where $i+1$ is the the successor of i, the lifting $X_{i+1} \to A$ exists. If i is a limit number, then from the univsersal property of colimits we see there will exist $\underset{j<i}{\operatorname{colim}} X_j \to A$. And then since $\underset{j<i}{\operatorname{colim}} X_j \to X_i$ belongs to $l(\mathcal{F})$, the lifting $X_i \to A$ exists. Finally there will be some $\underset{i \in I}{\operatorname{colim}} X_i \to A$ making the diagram commutative. $\square$

Corollary A.1.6. *In a category $\mathcal{C}$, for any class of morphisms $\mathcal{F}$, $r(\mathcal{F})$ is closed under retraction, pullbacks, products and the dual process of transfinite compositions especially finite compositions.*

Proof. Apply Theorem A.1.5 to $\mathcal{C}^{op}$. $\square$

Fact A.1.7. In a category $\mathcal{C}$ there are two classes of morphisms $\mathcal{F}$ and $\mathcal{F}'$, then

(1) $\mathcal{F} \subseteq r(\mathcal{F}') \Leftrightarrow \mathcal{F}' \subseteq l(\mathcal{F})$

(2) $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow l(\mathcal{F}') \subseteq l\mathcal{F}$.

(3) $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow r(\mathcal{F}') \subseteq r\mathcal{F}$.

(4) $r(\mathcal{F}) = r \circ l \circ r(\mathcal{F})$.

(5) $l(\mathcal{F}) = l \circ r \circ l(\mathcal{F})$.

Proof. We only prove the property of (4). Since all morphisms in $\mathcal{F}$ have the LLP wrt $r(\mathcal{F})$, then $\mathcal{F} \subseteq l \circ r(\mathcal{F})$. This implies $r \circ l \circ r(\mathcal{F}) \subseteq r(\mathcal{F})$. Replacing $\mathcal{F}$ by $r(\mathcal{F})$, we see it's clear that $r(\mathcal{F}) \subseteq r \circ l(r(\mathcal{F}))$. $\square$

Definition A.1.8. A *weak factorization system* in a category $\mathcal{C}$ is a couple $(\mathcal{F}, \mathcal{G})$ of classes of morphisms satisfying

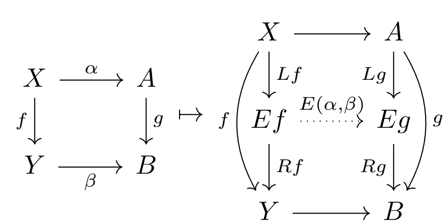
(1) both $\mathcal{F}$ and $\mathcal{G}$ are closed under retraction.

(2) $\mathcal{F} \subseteq l(\mathcal{G}) (\Leftrightarrow \mathcal{G} \subseteq r(\mathcal{F})$.

(3) any morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a factorization $f = p \circ i$ where $i \in \mathcal{F}$ and $p \in \mathcal{G}$.

In the most cases, we may require the factorization $f = p \circ i$ to be functorial and we will explain what's the meaning of *functorial factorization system*. In the factorization $f = p \circ i$, we use Rf and Lf to denote p and i respectively.

Definition A.1.9. In a category $\mathcal{C}$, a *functorial factorization system* is a weak factorization system with a functor $\mathcal{C}^2 \to \mathcal{C}^3$ from the category of arrows in $\mathcal{C}$ to the category of composable pairs of arrows in $\mathcal{C}$.



This definition actually means the replacement functors L and R are functorial. The reason why we often require this property is practical. The method Quillen uses to construct a model category is called *small object arguments*. From this method, we always obtain a functorial factorization system, since every step of this method is functorial. Now let us introduce this method which can also be found in [14, Proposition 2.1.9], [31, Theorem 2.1.14] and [55, Theorem 12.2.2].

Definition A.1.10. Given a cardinal κ, a non-empty partially ordered set E is *κ-filtered* if for any family of its elements x_j indexed by J with $|J| < \kappa$, then there exists an element $x \in E$ such that $x_j \leq x$ for all $j \in J$.

Theorem A.1.11 (Small Object Argument)**.** *Let $\mathcal{C}$ be a locally small category with small colimits, equipped with a small set of morphisms $\mathcal{F}$. If there exists a cardinal κ such that for any element $i : K \to L$ in $\mathcal{F}$, the functor*

$$\mathrm{Hom}_{\mathcal{C}}(K, -) : \mathcal{C} \to \mathbf{Sets}$$

commutes with colimits indexed by κ-filtered well-ordered sets, then the couple $(l \circ r(\mathcal{F}), r(\mathcal{F}))$ forms a functorial factorization system and $l \circ r(\mathcal{F})$ is the smallest saturated class containing $\mathcal{F}$.

Proof. Suppose κ exists and $\lambda \geq \kappa$. Given any morphism $f : X \to Y$ in $\mathrm{Mor}(\mathcal{C})$, let D_0 be the class consisting of all commutative diagrams

$$\begin{array}{ccc} K & \longrightarrow & X = Z_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f = f_0} \\ L & \longrightarrow & Y \end{array}$$

with $i \in \mathcal{F}$. Then let Z_1 be the pushout of

$$\begin{array}{ccccc} \coprod_{D_0} K & \longrightarrow & X = Z_0 & & \\ \downarrow & & \downarrow & \searrow^{f_0} & \\ \coprod_{D_0} L & \longrightarrow & Z_1 & \overset{\exists!}{\underset{f_1}{\dashrightarrow}} & Y \\ & \searrow & & \nearrow & \end{array}$$

If $i+1$ is the successor of i, using this method to obtain Z_{i+1} and $f_{i+1} : Z_{i+1} \to Y$ from Z_i and f_i. If i is a limit number, define $Z_i = \operatorname{colim}_{j<i} Z_j$ and this will induce a morphism $f_i : Z_i \to Y$ from f_j's. Finally we have the following factorization

$$\begin{array}{ccccccccccc} X = Z_0 & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_i = \operatorname{colim}_{j<i} Z_j & \longrightarrow & \cdots & \longrightarrow & Z_\lambda \\ {\scriptstyle f_0}\downarrow & \swarrow_{f_1} & & & \swarrow_{f_i} & & & & \swarrow_{f_\lambda} & & \\ Y & & & & & & & & & & \end{array}$$

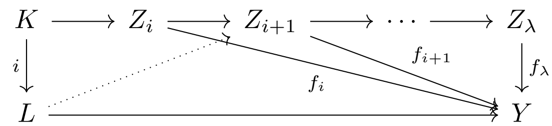
with $g : X = Z_0 \to Z_\lambda$ the composite along the top row.

We prove $g \in l \circ r(\mathcal{F})$ and $f_\lambda \in r(\mathcal{F})$.

Given a lifting problem

$$\begin{array}{ccc} K & \longrightarrow & Z_\lambda \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ L & \longrightarrow & Y \end{array}$$

with $i \in \mathcal{F}$. Since $\mathrm{Hom}_{\mathcal{C}}(K, -)$ commutes with λ-colimits, $K \to Z_\lambda = \operatorname{colim}_{j<\lambda} Z_j$ factors through some Z_i.

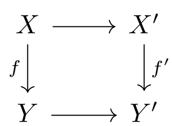


But from the definition of Z_{i+1}, we see $i : K \to L$ here belongs to D_i, which means there exists a lifting $L \to Z_{i+1}$ whose composition with $Z_{i+1} \to Z_\lambda$ gives the lifting $L \to Z_\lambda$.

Next we should prove $g \in l \circ r(\mathcal{F})$. It's obvious to see $\mathcal{F} \subseteq l \circ r(\mathcal{F})$. From Theorem A.1.5,$l \circ r(\mathcal{F})$ is saturated. But $g : X \to Z_\lambda$ is obtained by "attaching cells" of $\coprod_{D_i} K \to \coprod_{D_i} L$. Therefore it's clear $g \in l \circ r(\mathcal{F})$.

The statements in the previous paragraph obviously imply that the smallest saturated class of $\mathcal{F}$ is contained in $l \circ r(\mathcal{F})$. On the other hand, assume $f : X \to Y$ belonging to $l \circ r(\mathcal{F})$, then $f = p \circ i$ where $i \in l \circ r(\mathcal{F})$ and $p \circ r(\mathcal{F})$ which we haved proved above. Note that from the proof, i is in the smallest saturated class of $\mathcal{F}$. Since f has the LLP wrt p, from Lemma A.1.2 f is a retract of i hence belonging to the smallest saturated class of $\mathcal{F}$ by Definition A.1.3.

Finally we should prove this factorization talked above is functorial. Given a commutative diagram



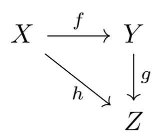
then we have classed D_0 and D_0' for f and f' respectively. But it's clear there is a map $D_0 \to D_0'$ via compositions and this will induce a map $Z_1 \to Z_1'$. And finally we will obtain $Z_\lambda \to Z_\lambda'$. Since in every step of this process the morphism is induced by the universal property, it's clear the final map $Z_\lambda \to Z_\lambda'$ is functorial. □

Now let us introduce the concept of model categories.

Definition A.1.12. A *model category* $\mathcal{M}$ has three classes of morphisms which are denoted by Cof, Fib and $\mathcal{W}$, and are called cofibtrations, fibrations and weak equivalences respectively. Moreover it satisfies the following axioms

(M1) $\mathcal{M}$ has all finite limits and colimits.

(M2) In the commutative diagram:



If any two of the three morphisms f, g and $h = g \circ f$ are weak equivalences, then so is the other. This property is called "two out of three".

(M3) Cof, Fib and $\mathcal{W}$ are closed under retraction.

(M4) $Cof \boxslash (Fib \cap \mathcal{W})$ and $(Cof \cap \mathcal{W}) \boxslash Fib$.

(M5) Every morphism $f : X \to Y$ in $\mathcal{M}$ can be factored as $f = p \circ i = q \circ j$ such that $i \in Cof \cap \mathcal{W}$, $p \in Fib$, $j \in Cof$ and $q \in Fib \cap \mathcal{W}$.

Note that axioms (M4) and (M5) actually mean $(Cof, Fib \cap \mathcal{W})$ and $(Cof \cap \mathcal{W}, Fib)$ form two weak factorization systems. Morphisms in $Cof \cap \mathcal{W}$ (resp. $Fib \cap \mathcal{W}$) are called *trivial cofibrations* (resp. *trivial fibrations*). In a model category we use $\emptyset$ and $*$ to denote its initial object and terminal object respectively. An object $X \in \mathcal{M}$ is *cofibrant* (resp. *fibrant*) if $\emptyset \to X$ (resp. $X \to *$) is a cofibration (resp. fibration). Sometimes we write $\bullet \xrightarrow{\sim} \bullet$ for a weak equivalence, $\bullet \rightarrowtail \bullet$ for a cofibration and $\bullet \twoheadrightarrow \bullet$ for a fibration.

Example A.1.13. $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ denotes the full subcategory of $\mathbf{Ch}(R)$ consisting of all complexes of left R-modules such that $C_n = 0$ for $n < 0$. Then $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ is a model category. Actually $\mathbf{Ch}(R)$ is also a model category whose structure is similar to the former.

$f : C_\bullet \to D_\bullet$ in $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ is a weak equivalence (also called quasi-isomorphism) if it is a homology isomorphism which means it induces isomorphisms between homology groups. f is a fibration if $f_n : C_n \to D_n$ for $n > 0$ are all surjective. Then cofibrations in $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ can be defined as those maps having the LLP wrt all trivial fibrations. In $\mathbf{Ch}_{\geq \mathbf{0}}(R)$ a complex is cofibrant if and only if all modules in it are projective.

Dually in the category of cochains $\mathbf{Ch}^{\geq \mathbf{0}}(R)$ there is a model category structure as well, in which weak equivalences are quasi-isomorphisms, cofibrations are monomorphisms for positive terms and fibrations are epimorphisms whose kernels consist of injective modules.

Therefore there are two model structures in $\mathbf{Ch}(R)$. One is the projective model structure and the other is the injecture one. In the projective model category of $\mathbf{Ch}(R)$, weak equivalences are quasi -isomorphisms, fibrations are epimorphisms and cofibrations are those maps having the left lifting property with respect to all trivial fibrations. Note that in this case even though for every cofibrant chain complex $A_\bullet$, A_n will be a projective R-module for all n and conversely any bounded below complex of projective R-modules is cofibrant, there may exist unbounded complex of projective R-modules, which isn't cofibrant [31, Remark 2.3.7].

Example A.1.14. The category **Top** of topological spaces is a model category with weak equivalences being *weak homotopy equivalences* which induce isomorphisms on homotopy groups, fibrations being *Serre fibrations* and cofibrations being those having the LLP wrt all trivial fibrations. Every *relative CW-complex* will be a cofibration. Details can be found in [52] or [33, Lecture 02].

Example A.1.15. Objects in the category Δ consist of all $[n]$ for $n \geq 0$ where $[n] = \{0, 1, \cdots, n\}$. And morphisms are non-decreasing functions. A *simplicial object* of a category $\mathcal{C}$ is defined to be a functor $\Delta^{op} \to \mathcal{C}$. Therefore a *simplicial set* is a functor $\Delta^{op} \to \mathbf{Set}$. The category of simplicial sets is denoted by $s\mathbf{Set} = \mathbf{Set}^{\Delta^{op}}$. $s\mathbf{Set}$ is a model category with weak equivalences being morphisms which induce weak homotopy equivalences at the level of *geometric realization*, fibrations being *Kan fibrations* and cofibrations being injective maps. For detailed references we recommend [14] and [23].

A model category is *cofibrantly generated* if its factorization systems are obtained by applying small object argument to some small set of cofibrations and trivial cofibrations. All examples above are cofibrantly generated.

A.2 Homotopy Theory

Before introducing the homotopy theory for a model category $\mathcal{M}$, we want to study the internal structure of $\mathcal{M}$ first.

Lemma A.2.1. *For a weak factorization system (see Definition A.1.8) $(\mathcal{F}, \mathcal{G})$, $\mathcal{F} = l(\mathcal{G})$ and $\mathcal{G} = r(\mathcal{F})$.*

Proof. We only need to prove $\mathcal{F} = l(\mathcal{G})$ since the other can be proved in the opposite category $\mathcal{C}^{op}$. Assume $f \in l(\mathcal{G})$ and $f = p \circ i$ where $i \in \mathcal{F}$, $p \in \mathcal{G}$. f has the LLP wrt p and from Lemma A.1.2 we conclude f is a retraction of i which means $f \in \mathcal{F}$. □

Corollary A.2.2. *In a model category $\mathcal{M}$,*

(1) the cofibrations are exactly those maps having the LLP wrt all trivial fibrations

(2) *the trivial cofibrations are exactly those maps having the LLP wrt all fibrations*

(3) *the fibrations are exactly those maps having the RLP wrt all trivial cofibrations*

(4) *the trivial fibrations are exactly those maps having the RLP wrt all cofibrations*

Proof. Apply Lemma A.2.1. □

This corollary means for a given category $\mathcal{M}$ if a certain model category structure exists, then it can be characterized by $\mathcal{W}$ and Fib or $\mathcal{W}$ and Cof. But if we only know Cof and Fib, then we will also know $Cof \cap \mathcal{W}$ and $Fib \cap \mathcal{W}$ which are characterized by their lifting properties. And we can define a weak equivalence to be a composition $p \circ i$ where $i \in Cof \cap \mathcal{W}$ and $p \in Fib \cap \mathcal{W}$. Therefore $\mathcal{M}$ can also be determined by Cof and Fib. There is a refined version for Corollary A.2.2.

Proposition A.2.3. *In a model category $\mathcal{M}$,*

(1) *a cofibration is a weak equivalence iff it has the LLP wrt all fibrations between fibrant objects*

(2) *a fibration is a weak equivalence iff it has the RLP wrt all cofibrations between cofibrant objects.*

Proof. We only prove (1) since the second one can be proved in $\mathcal{M}^{op}$. "⇒" is clear and thus we prove the part of "⇐". Assume a cofibration $u : A \to B$ has the LLP wrt all fibrations between fibrant objects. First we choose a fibrant replacement $j : B \to B'$ where j is a trivial cofibration and B' is fibrant. Then we factor $j \circ u$ as

$$j \circ u : A \xrightarrow{i} A' \xrightarrow{p} B'$$

where i is a trivial cofibration and p is fibration which imply A is especially fibrant. Then we have the square

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & A' \\ {\scriptstyle u}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\;j\;]{} & B' \end{array}$$

$h \circ u = i$ and $p \circ h = j$ are all isomorphisms in the homotopy category $\mathrm{Ho}(\mathcal{M})$. Hence in $\mathrm{Ho}(\mathcal{M})$, h has a left inverse and a right inverse, thus an isomorphism. Then according to Corollary A.2.23, h is a weak equivalence and therefore u is a weak equivalence. □

We also has another characterization for $\mathcal{M}$ which is useful when comparing with different models for ∞-category. We advise readers to read this proposition after reading all contents of this section, since in this proof we will use theorems below freely.

Proposition A.2.4. *For a model category $\mathcal{M}$, its model category structure is determined by cofibrations and fibrant objects or fibrations and cofibrant object.*

Proof. Since two statements are dual, it's enough to prove the first one. For any object $X \in \mathrm{Ob}(\mathcal{M})$, we have the decomposition $\emptyset \to X' \xrightarrow{p_X} X$ where X' is cofibrant and p_X is a trivial fibration. For any map $u : X \to Y$ we have the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y' \\ \downarrow & \overset{u'}{\nearrow} & \downarrow{\scriptstyle p_Y} \\ X' & \xrightarrow[u \circ p_X]{} & Y \end{array}$$

which permits the existence of the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\;p_X\;} & X \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow[\;p_Y\;]{} & Y \end{array}$$

Note that u is weak equivalence iff u' is a weak equivalence. Therefore weak equivalences between cofibrant objects will determine this model category structure, since from Cof we know $Fib \cap \mathcal{W}$ and $p_X,\ p_Y$ are all trivial fibrations. u' is a weak equivalence iff it's an isomorphism in $\mathrm{Ho}(\mathcal{M})$. By Yoneda's lemma, it's a weak equivalences iff for any other object A, $u'^{*} : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(Y', A) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A)$ is an isomorphism. But the full subcategory of $\mathrm{Ho}(\mathcal{M})$ consisting of fibrant objects is equivalent to $\mathrm{Ho}(\mathcal{M})$. Therefore we could suppose A is fibrant. But by Propodition A.2.18 this means

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A) = [X', A] = \mathrm{Hom}_{\mathcal{M}}(X', A)/\sim$$

where the equivalence relation is right homotopy or left homotopy. Here we focus on the left homotopy. But we can factor $(\mathrm{id}, \mathrm{id}) : X' \coprod X' \to X'$ as

$$X' \coprod X' \xrightarrow{i} X' \otimes I \xrightarrow{p} X'$$

where i is a cofibration and p is a trivial fibration. This factorization will make the cylinder object $X' \otimes I$ for the left homotopy relation fixed, which means this left homotopy relation is totally determined by cofibraions and trivial fibrations. Hence we conclude $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X', A)$ is determined. And finally weak equivalences between cofibrant objects are determined. □

For a model category $\mathcal{M}$, its *homotopy theory* or *homotopy category* is defined to be $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$ the localization respect to weak equivalences. In the following our main task is to study structures of $\mathrm{Ho}(\mathcal{M})$ in detail.

Definition A.2.5. A *path object* for $Y \in \mathrm{Ob}(\mathcal{M})$ is a commutative diagram:

$$\begin{array}{ccc} & & Y^I \\ & \overset{s}{\underset{\sim}{\nearrow}} & \Big\downarrow {\scriptstyle p=(p_0,p_1)} \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

where s is a weak equivalence, $\Delta = (\mathrm{id}_Y, \mathrm{id}_Y)$ and (p_0, p_1) is a fibration.

Always we simply use the symbol Y^I to denote a path object. According to the axiom (M5) of model categories, there is a natural path object for Y such that s will be a trivial cofibration.

Definition A.2.6. $f, g : X \to Y$ are two maps in $\mathcal{M}$. A *right homotopy* between f and g is a commutative diagram:

$$\begin{array}{ccccc} & & Y^I & & \\ & \overset{h}{\nearrow} & \Big\downarrow {\scriptstyle p} & \overset{s}{\underset{\sim}{\nwarrow}} & \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xleftarrow[\Delta]{} & Y \end{array}$$

The right part of the diagram above is a path object. We denote this relation by $f \simeq_r g$.

Example A.2.7. In $\mathbf{Ch}_{\geq 0}(R)$, the concept of chain homotopies is a special case of right homotopies.

Given a chain complex C , the path object is defined to be C^I such that $C^I_n = C_n \oplus C_n \oplus C_{n+1}$ for $n > 0$ and

$$C^I_0 = \{(x, y, z) \in C_0 \oplus C_0 \oplus C_1 | (x - y) + \partial_1(z) = 0\}$$

$\partial_n(x, y, z) = (\partial_n(x), \partial_n(y), (-1)^n(x - y) + \partial_{n+1}(z))$. We claim C^I is a path object.

We define a new chain complex C' as follows: $C'_n = C_n \oplus C_{n+1}$ for $n > 0$ and

$$C'_0 = \{(x, z) \in C_0 \oplus C_1 | x + \partial_1(z) = 0\}$$

$\partial_n(x, z) = (\partial_n(x), (-1)^n x + \partial_{n+1}(z))$. If $\partial_n(x, z) = 0$, then $\partial_n(x) = 0$ and $(-1)^n x + \partial_{n+1}(z) = 0$. $x = (-1)^{n+1}\partial_{n+1}(z)$, $\partial_{n+1}((-1)^{n+1}, 0) = ((-1)^{n+1}\partial_{n+1}z, (-1)^{n+1}(-1)^{n+1}z) = (x, z)$. Hence, $C' \to 0$ is a trivial fibration. There is a chain map $\alpha : C^I \to C'$, $\alpha(x, y, z) = (x - y, z)$, which is an epimorphism and $\ker \alpha \cong C$. . Then we have the following pullback diagram:

$$\begin{array}{ccc} C^I & \xrightarrow{\alpha} & C' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ C \oplus C & \xrightarrow{\beta} & C \end{array}$$

where $p(x, y, z) = (x, y)$, $p'(x, z) = x$ and $\beta(x, y) = x - y$. Obviously p' is a fibration. Hence p is also a fibration. We have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{s} & C^I & \xrightarrow{\alpha} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \\ 0 & \longrightarrow & C & \xrightarrow{\Delta} & C \oplus C & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

where $s(x) = (x, x, 0)$. There is a long exact sequence

$$\cdots \longrightarrow 0 = H_{n+1}(C') \longrightarrow H_n(C) \longrightarrow H_n(C^I) \longrightarrow 0 = H_n(C') \longrightarrow \cdots$$

Hence $H_n(C) \cong H_n(C)$ and s is a weak equivalence.

For any two chain map $f, g : D \to C$ such that there is a usual chain homotopy $t : f \simeq g$, $t_n : D_n \to C_{n+1}$ and $\partial_{n+1} t_n + t_{n-1}\partial_n = f - g$. We define $h : C \to C^I$, $h_n(x) = (f_n(x), g_n(x), (-1)^{n+1} t_n(x))$. It's actually a chain map. $\partial h(x) = (\partial f(x), \partial g(x), (-1)^{n+1}(f(x) - g(x)) + (-1)^{n+1}\partial t(x)) = (\partial f(x), \partial g(x), (-1)^n t\partial(x)) = h\partial(x)$. $p \circ h = (f, g)$. Hence, h is a right homotopy from f to g.

Conversely, if h is a right homotopy from f to g, we write h as (f, g, t). Because h is a chain map, $(-1)^n(f(x) - g(x)) + \partial t(x) = t\partial(x)$. Then $t'_n = (-1)^{n+1} t_n$ is a chain homotopy from f to g in the usual sense.

Definition A.2.8. A *cylinder object* for $X \in \mathcal{M}$ is a commutative diagram:

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ {\scriptstyle (i_0, i_1)}\downarrow & \nearrow{\scriptstyle \sim\, \sigma} & \\ X \otimes I & & \end{array}$$

where $\nabla = (\mathrm{id}_X, \mathrm{id}_X)$, i is a cofibration and s is a weak equivalence. For any maps $f, g : X \to Y$, a *left homotopy* $h : f \simeq_l g$ is defined to be the following commutative diagram:

$$\begin{array}{ccccc} Y & \xleftarrow{(f,g)} & X \coprod X & \xrightarrow{\nabla} & X \\ & \nwarrow{\scriptstyle h} & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \sim\, \sigma} & \\ & & X \otimes I & & \end{array}$$

In **Top** for CW-complexes, the concept of homotopies is a special case of left homotopies.

Example A.2.9. We assume X is a CW-complex and $X \otimes I = X \times I$ where $I = [0, 1]$. Then X is a strong deformation retract of $X \times I$ and σ is especially a weak homotopy equivalence. $(X \coprod X, X \times I)$ is a relative CW-complex (see [27, Theorem A.6]). Hence $i : X \coprod X \hookrightarrow X \times I$ is a cofibration.

Lemma A.2.10. *Let $\mathcal{M}$ be a model category,*

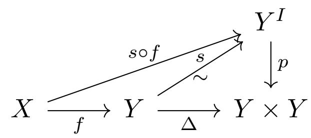
(1) if Y is fibrant, then the relation of right homotopies in $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is an equivalence relation.

(2) if X is cofibrant, then the relation of left homotopies in $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is an equivalence relation.

Proof. Axioms of a model category are all dual descriptions, which means $\mathcal{M}^{op}$ is also a model category with cofibrations becoming fibrations and fibrations becoming cofibrations. Hence we can just prove the first statement.

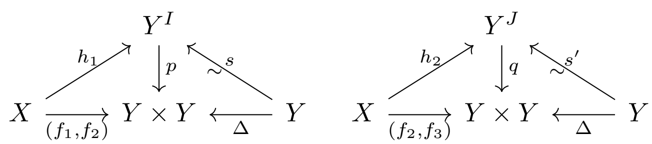
Assume Y is fibrant, which means $Y \to *$ is a fibration. In $\mathcal{M}$, for any object X, $X \times *$ is canonically isomorphic with X. Isomorphisms are both trivial cofibrations and trivial fibrations. Then the fact fibrations are preserved under products implies $X \times Y \to X$ is a fibration. Hence $pr_0, pr_1 : Y \times Y \to Y$ are fibratsions. For any path object Y^I with a fibration $p : Y^I \to Y \times Y$, $pi = pr_i \circ p$ is a fibration for $i = 0, 1$. Moreover, $p \circ s = \Delta$, $p_i \circ s = pr_i \circ \Delta = \mathrm{id}_Y$. Then p_i is a trivial fibration.

For any $f : X \to Y$, the following diagram proves $f \simeq_r f$.

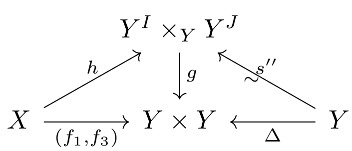


For any map $f, g : X \to Y$ and $h : f \simeq_r g$. There is an isomorphism $u = (pr_1, pr_0) : Y \times Y \to Y \times Y$. If $p : Y^I \to Y \times Y$ is the path object for the right homotopy $h : f \simeq_r g$, then $u \circ p$ is the path object for $h : g \simeq_r f$.

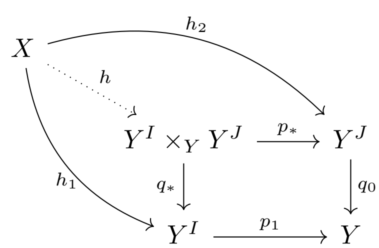
$f_1, f_2, f_3 : X \to Y$ and $h_1 : f_1 \simeq_r f_2$, $h_2 : f_2 \simeq_r f_3$.



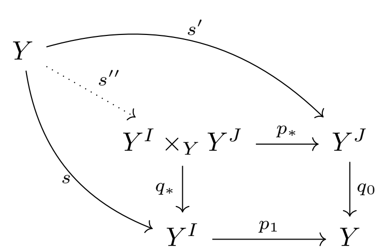
We prove there is a right homotopy:



From h_1, h_2 we know $p_0 h_1 = f_1, p_1 h_1 = f_2 = q_0 h_2, q_1 h_2 = f_3$. Then we have the following pullback diagram:



where q_0 is a trivial fibration $\Rightarrow$ q_* is also a trivial fibration.



$q_* \circ s'' = s$ is a weak equivalence $\Rightarrow s''$ is a weak equivalence. Now we only need to find the suitable g. We prove the following diagram is a pullback:

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ {\scriptstyle (q_*, q_1 p_*)}\downarrow & & \downarrow{\scriptstyle (q_0, q_1)} \\ Y^I \times Y & \xrightarrow[p_1 \times id_Y]{} & Y \times Y \end{array}$$

$(p_1 \times \mathrm{id}_Y) \circ (q_*, q_1 p_*) = (p_1 q_*, q_1 p_*) = (q_0 p_*, q_1 p_*) = (q_0, q_1) \circ p_*$. The diagram is commutative. Given $(u, v) : Z \to Y^I \times Y, w : Z \to Y^J$ such that $(p_1 \times \mathrm{id}_Y) \circ (u, v) = (q_0, q_1) \circ w$. Then $p_1 u = q_0 w, v = q_1 w$. From $p_1 u = q_0 w$, there is a unique $\theta : Z \to Y^I \times_Y Y^J$ such that $q_* \theta = u, p_* \theta = w$. $q_1 p_* \theta = q_1 w = v$. Hence, $(q_*, q_1 p_*) \circ \theta = (u, v)$. This proves the diagram above is actually a pullback.

We let $g = (p_0 \times \mathrm{id}_Y) \circ (q_*, q_1 p_*) = (p_0 q_*, q_1 p_*)$. $(q_*, q_1 p_*)$ is the pullback of $q = (q_0, q_1) \Rightarrow (q_*, q_1 p_*)$ is a fibration. p_0, id_Y are fibrations $\Rightarrow p_0 \times \mathrm{id}_Y$ is a fibration. Hence g is a fibration.

$g \circ h = (p_0 q_*, q_1 p_*) \circ h = (p_0 q_* h, q_1 p_* h) = (p_0 h_1, q_1 h_2) = (f_1, f_3)$ and $g \circ s'' = (p_0 q_*, q_1 p_*) s'' = (p_0 q_* s'', q_1 p_* s'') = (p_0 s, q_1 s') = (\mathrm{id}_Y, \mathrm{id}_Y) = \Delta$. ☐

Lemma A.2.11.

(1) If Y is fibrant, $X \otimes I$ is a fixed cylinder object for X and $f, g : X \to Y$ are right homotopic, then there is a left homotopy:

$$\begin{array}{ccc} Y & \xleftarrow{(f,g)} & X \coprod X \\ & {\scriptstyle H}\nwarrow & \downarrow{\scriptstyle i} \\ & & X \otimes I \end{array}$$

(2) If X is cofibrant, Y^I is a fixed path object for Y, and $f, g : X \to Y$ are left homotopic, then there is a right homotopy:

$$\begin{array}{ccc} & & Y^I \\ & {\scriptstyle H}\nearrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

Proof. The two statements are dual and we prove the first one. Given the right homotopy $h : f \simeq_r g$:

$$\begin{array}{ccccc} & & Y^I & & \\ & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} & \nwarrow{\scriptstyle \sim\, s} & \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xleftarrow[\Delta]{} & Y \end{array}$$

and the fixed cylinder object:

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \sim\, \sigma} & \\ X \otimes I & & \end{array}$$

Note that Y is fibrant $\Rightarrow p_0 : Y \times Y \to Y$ is a trivial fibration.

$$\begin{array}{ccccc} X \coprod X & \xrightarrow{(sf,h)} & Y^I & \xrightarrow{p_1} & Y \\ {\scriptstyle i}\downarrow & {\scriptstyle \theta}\nearrow & \downarrow{\scriptstyle p_0} & & \\ X \otimes I & \xrightarrow[f\sigma]{} & Y & & \end{array}$$

$p_0 \circ (sf, h) = (p_0 sf, p_0 h) = (f, f) = f \sigma i$. Let $H = p_1 \circ \theta$. $H \circ i = p_1 \theta i = p_1 \circ (sf, h) = (p_1 sf, p_1 h) = (f, g)$. ☐

Corollary A.2.12. *If X is cofibrant and Y is fibrant, then for any maps $f, g : X \to Y$, $f \simeq_r g \Leftrightarrow f \simeq_l g$ for a fixed $X \otimes I \Leftrightarrow f \simeq_r g$ for a fixed $Y^I \Leftrightarrow f \simeq_l g$.*

In $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ where X is cofibrant and Y is fibrant, the two notions of right homotopy and left homotopy coincide and we simply use the symbol $f \simeq g$ to denote this homotopy relation. The homotopy class of maps between X and Y is denoted by $[X, Y] = \mathrm{Hom}_{\mathcal{M}}(X, Y)/\sim$. If $\mathcal{M}_c$ and $\mathcal{M}_f$ are the full subcategories of $\mathcal{M}$ with all objects cofibrant and fibrant respectively, then we can define a functor $[-,-] : \mathcal{M}_c^{op} \times \mathcal{M}_f \to \mathbf{Set}$. The fact that $[-,-]$ is well defined and actually a functor can be proved using the following lemma.

Lemma A.2.13. *For any objects $X, Y \in Ob(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if $f \simeq_l g$ then for any morphism $t : Y \to Y'$, $tf \simeq_l tg$.

(2) if $f \simeq_r g$ then for any morphism $s : X' \to X$, $fs \simeq_r gs$.

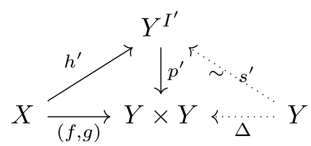
This lemma is trivial. In general, given two arbitrary morphisms $f, g : X \to Y$ and $t : Y \to Y'$, we can't conclude $tf \simeq_r tg$ from $f \simeq_r g$. But there is a weaker theorem, which states that in $\mathcal{M}_f$, $f \simeq_r g$, for arbitrary $t : Y \to Y'$ there exsits a trivial fibration especially a weak equivalence $u : X' \to X$ such that $tfu \simeq_r tgu$.

Lemma A.2.14. *For any objects $X, Y \in \mathrm{Ob}(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if Y is fibrant and $f \simeq_l g$, then for any morphism $s : X' \to X$, $fs \simeq_l gs$.

(2) if X is cofibrant and $f \simeq_r g$ then for any morphism $t : Y \to Y'$, $tf \simeq_r tg$.

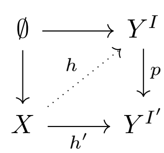
Proof. We only prove (2). Given a right homotopy $h' : f \simeq_r g$,



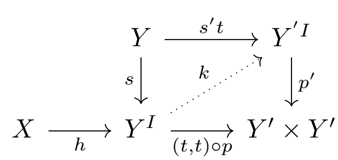
Decompose s' as

$$Y \xrightarrow{s} Y^I \xrightarrow{p} Y^{I'}$$

such that s is trivial cofibration and p is a fibration. $p' \circ p$ is a fibration and Y^I is thus a path object. Since $p \circ s = s'$ is a weak equivalence, p is a trivial fibration. Then the following lifting problem has a solution since X is cofibrant.



Then $h : f \simeq_r g$ with the map $s : Y \to Y^I$ being a trivial cofibration. Here we simply write $p : Y^I \to Y \times Y$ for $p' \circ p$ above and use p', s' to denote maps $Y'^I \to Y' \times Y'$ and $Y' \to Y'^I$ respectively.



$k \circ h : tf \simeq_r tg$. ☐

For any two objects $X, Y \in \mathrm{Ob}(\mathcal{M})$, $\pi^l(X, Y)$ (resp. $\pi^r(X, Y)$) denotes the quotient set $\mathrm{Hom})_{\mathcal{M}}(X, Y)/\sim$ where the equivalence relation is generated by left (resp. right) homotopy. $f \sim_l g$ in $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ if there is a long sequence of left homotopy connecting them, which means $f \simeq_l f_1 \simeq_l f_2 \simeq_l \cdots \simeq_l g$.

Corollary A.2.15. *For any objects $X, Y \in \mathrm{Ob}(\mathcal{M})$ and morphisms $f, g : X \to Y$,*

(1) if Y is fibrant, then the composition $\pi^l(X', X) \times \pi^l(X, Y) \to \pi^l(X', Y)$ is well defined.

(2) if X is cofibrant, then the composition $\pi^r(X, Y) \times \pi^r(Y, Y') \to \pi^r(X, Y')$ is well defined.

We can define the homotopy category $\pi\mathcal{M}_c$ (resp. $\pi\mathcal{M}_f$) to be the quotient category of $\mathcal{M}_c$ (resp. $\mathcal{M}_f$) where the equivalence relation is the right (resp. left) homotopy relation and $\mathrm{Hom}_{\pi\mathcal{M}_c}(X, Y) = \pi^r(X, Y)$. Then from Corollary A.2.15 this definition is well defined. $\pi\mathcal{M}_{cf}$ is in the usual sense with $\mathrm{Hom}_{\pi\mathcal{M}_{cf}}(X, Y) = [X, Y]$ since $X,\ Y$ are both cofibrant and fibrant.

Next we prove the modern version of Whitehead's theorem whose classical version says a weak homotopy equivalence between CW-complexes is homotopy equivalence.

Theorem A.2.16 (Whitehead)**.** *If X, Y are both cofibrant and fibrant, then every weak equivalence $f : X \to Y$ is a homotopy equivalence.*

Proof. According to (M5) of model categories, $f : X \xrightarrow{i} Z \xrightarrow{p} Y$ where i is a trivial cofibration and p is trivial fibration. We prove Z is both cofibrant and fibrant first.

Let $A \to B$ be a trivial cofibration:

$$\begin{array}{ccccc} A & \longrightarrow & Z & \xrightarrow{p} & Y \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B & \longrightarrow & * & \longrightarrow & * \end{array}$$

where Y is fibrant and p is trivial fibration. Hence Z is fibrant. Dually, Z is cofibrant.

If p and i are homotopy equivalences, there are $q : Y \to Z, j : Z \to X$ such that $pq \simeq \mathrm{id}_Y, qp \simeq \mathrm{id}_Z, ij \simeq \mathrm{id}_Z, ji \simeq \mathrm{id}_X$. Then

$$fjq = pijq \simeq p\mathrm{id}_Z q \simeq pq \simeq \mathrm{id}_Y$$

and

$$jqf = jqpi \simeq j\mathrm{id}_Z i \simeq ji \simeq \mathrm{id}_X$$

which means f is a homotopy equivalence as well. Now we need to prove every trivial fibration is a homotopy equivalence and it's dual to prove every trivial cofibration is a homotopy equivalence. Hence we can just assume f is a trivial fibration.

Since Y is cofibrant,

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \overset{g}{\nearrow} & \downarrow f \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

then $fg = \mathrm{id}_Y$, and we only need to prove $gf \simeq \mathrm{id}_X$.

Given a cylinder object,

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ i\downarrow & \overset{\sim\ \sigma}{\nearrow} & \\ X \otimes I & & \end{array}$$

there exists the following homotopy $h : gf \simeq_l \mathrm{id}_X$ since f is a trivial fibration and i is cofibration.

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(gf, \mathrm{id}_X)} & X \\ i\downarrow & \overset{h}{\nearrow} & \downarrow f \\ X \otimes I & \xrightarrow[f\sigma]{} & Y \end{array}$$

where $f \circ (gf, \mathrm{id}_X) = (fgf, g) = (f, f) = f\sigma i.$ $\square$

In **Top** every space is fibrant and every CW-complex is cofibrant. Hence the classical Whitehead's theorem tells us every weak homotopy equivalence between CW-complexes is actually a homotopy equivalence. And in $\mathbf{Ch}_{\geq 0}(R)$, every chain complex is fibrant and every chain complex with every term projective is cofibrant. Then, we can conclude two projective resolutions of a given R-module is chain homotopic.

The converse of Whitehead's theorem is also ture. If f is a homotopy equivalence then f is a weak equivalence. We will prove it later.

Now for every object X in $\mathcal{M}$, we find a suitable object RQX which is both cofibrant and fibrant and is weak equivalent with X. At first, we use the axiom (M5) to decompose $\emptyset \to X$ as $\emptyset \longrightarrow QX \xrightarrow{p_X} X$ such that QX is cofibrant and p_X is a trivial fibration. Then we decompose $QX \to *$ as $QX \xrightarrow{j_X} RQX \longrightarrow *$ such that j_X is a trivial cofibration and RQX is fibrant. In fact, because QX is cofibrant, RQX is cofibrant automatically. Hence RQX is both cofibrant and fibrant.

According to the descriptions above, we can find maps $X \xleftarrow{p_X} QX \xrightarrow{j_X} RQX$ for every $X \in \mathcal{M}$ such that:

(1) p_X is a trivial fibration, j_X is a trivial cofibration, QX is cofibrant and RQX is both cofibrant and fibrant.

(2) if X is cofibrant, $QX = X, p_X = \mathrm{id}_X$. And if QX is fibrant, $RQX = QX, j_X = \mathrm{id}_{QX}$.

From the condition (2), we can conclude $Q(RQX) = RQX$, $Q(QX) = QX$, $(RQ)(RQX) = RQX$, $(RQ)(QX) = RQX$.

If in our model category the weak factorization system is functorial, then we can find functorial replacement functors Q and R. But without this assumption the functorial property is only up to homotopy. Actually nearly all of model categories we meet are functorial and it's difficult to give an example not being functorial.

Lemma A.2.17. *For any map $f : X \to Y$, there will exist a commutative diagram:*

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

Moreover f_2 is unique up to homotopy.

Proof. Since QX is cofibrant and p_Y is a trivial fibration,

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & \overset{f_1}{\nearrow} & \downarrow{\scriptstyle p_Y} \\ QX & \xrightarrow[f \circ p_X]{} & Y \end{array}$$

Since RQY is fibrant and j_X is a trivial cofibration,

$$\begin{array}{ccc} QX & \xrightarrow{j_Y \circ f_1} & RQY \\ {\scriptstyle j_X}\downarrow & \overset{f_2}{\nearrow} & \downarrow \\ RQX & \longrightarrow & * \end{array}$$

Now we prove the uniqueness. If there is another commutative diagram (f_1', f_2'), we first prove $f_1 \simeq_l f_1'$. Given a cylinder object

$$\begin{array}{ccc} QX \coprod QX & \xrightarrow{\nabla} & QX \\ {\scriptstyle I}\downarrow & \overset{\sim\ \sigma}{\nearrow} & \\ QX \otimes I & & \end{array}$$

since i is a cofibration and p_Y is a trivial fibration,

$$\begin{array}{ccccc}
QX \coprod QX & \xrightarrow{(f_1, f_1')} & QY & \xrightarrow{j_Y} & RQY \\
\downarrow{\scriptstyle i} & \nearrow_{\text{(dotted lift to } QY)} & \downarrow{\scriptstyle p_Y} & & \\
QX \otimes I \xrightarrow[\sigma]{} QX & \xrightarrow[f p_X]{} & Y & &
\end{array}$$

where $p_Y \circ (f_1, f_1') = (p_Y f_1, p_Y f_1') = (f p_X, f p_X) = f p_X \sigma i$. From $f_1 \simeq_l f_1'$, we conclude $j_Y f_1 \simeq_l j_Y f_1'$. Hence $f_2 j_X \simeq_l f_2' j_X$. Since QX is cofibrant and RQY is fibrant, $f_2 j_X \simeq_r f_2' j_X$ and we have the following homotopy diagram:

$$\begin{array}{ccc}
 & & RQY^I \\
 & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\
QX & \xrightarrow[(f_2 j_X, f_2' j_X)]{} & RQY \times RQY
\end{array}$$

and since p is a fibration and j_X is a trivial cofibration

$$\begin{array}{ccc}
QX & \xrightarrow{h} & RQY^I \\
{\scriptstyle j_X}\downarrow & \overset{H}{\nearrow}\ \text{(dotted)} & \downarrow{\scriptstyle p} \\
RQX & \xrightarrow[(f_2, f_2')]{} & RQY \times RQY
\end{array}$$

Hence $f_2 \simeq f_2'$. □

According to the Lemma A.2.17, we can define a functor $RQ : \mathcal{M} \to \pi\mathcal{M}_{cf}$ such that $X \mapsto RQX$ and $f \mapsto [RQf] = [f_2]$. From the proof above we know that the functor $Q : \mathcal{M} \to \pi\mathcal{M}_c$ is also well defined, since according to Lemma A.2.11 (2) the left homotopy $f_1 \simeq_l f_1'$ can be changed into a right homotopy. Similarly there is a functor $R : \mathcal{M} \to \pi\mathcal{M}_f$ as well. Therefore we say the factorization system in a model category is functorial up to homotopy.

Now we can use the funtor $RQ : \mathcal{M} \to \pi\mathcal{M}_{cf}$ to obtain the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$. Objects of $\mathrm{Ho}(\mathcal{M})$ are the same as $\mathcal{M}$ and

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\pi\mathcal{M}_{cf}}(RQX, RQY) = [RQX, RQY]$$

From the uniqueness of f_2 up to homotopy, there is a functor

$$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M}),\ \gamma(X) = X,\ \gamma(f) = [RQf] = [f_2]$$

and the inclusion functor $\bar{\gamma} : \pi\mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M})$ is fully faithful and essentially surjective hence an equivalence between categories.

What's more, Whitehead's theorem tells us that if f is weak equivalence, then $\gamma(f)$ is an isomorphism. In the following, we will prove γ is actually a localization functor and $\mathrm{Ho}(\mathcal{M})$ is the category of fractions of $\mathcal{M}$ with respect to the set $\mathcal{W}$ of weak equivalences. Moreover $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence.

Proposition A.2.18. *If we assume X is cofibrant and Y is fibrant, then there is a bijection $[X, Y] \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = [RQX, RQY]$.*

Proof. Y is fibrant $\Rightarrow$ QY is fibrant. Then $QX = X$, $RQX = RX$, $RQY = QY$, $p_X = \mathrm{id}_X$,$j_Y = \mathrm{id}_{QY}$.

$\gamma : \mathrm{Hom}_{\mathcal{M}}(X, Y) \to [RQX, RQY] = [RX, QY]$. Given any map $f_2 : RX \to QY$, let $f_1 = f_2 \circ j_X$ and $f = p_Y \circ f_1$, then $\gamma(f) = [f_2]$ which means γ is surjective.

Next, we prove γ factors through $[X, Y]$. If $f, g : X \to Y$ and $f \simeq g$, we prove $f_2 \simeq g_2$. Since X is cofibrant and p_Y is a trivial fibration:

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & {\scriptstyle f_1} \nearrow\!\!\nearrow {\scriptstyle g_1} & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f/g]{} & Y \end{array}$$

where $f_1,\ g_1$ are liftings of $f,\ g$ respectively.

Given a left homotopy $h : f \simeq_l g$, we have a lifting

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(f_1, g_1)} & QY \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p_Y} \\ X \otimes I & \xrightarrow[h]{} & Y \end{array}$$

where $h \circ i = (f, g)$. Hence $f_1 \simeq g_1$. The same process will imply $f_2 \simeq f_2$. Then γ can be factored as

$$\mathrm{Hom}_{\mathcal{M}}(X, X) \longrightarrow [X, Y] \xrightarrow{\gamma'} [RQX, RQY] = [RX, QY]$$

Conversely if $f_2 \simeq g_2$, then $f_1 = f_2 \circ j_X \simeq g_2 \circ j_X = g_1$ and $f = p_Y \circ f_1 \simeq p_Y g_1 = g$. Hence γ is a bijection. $\square$

Theorem A.2.19. *$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is the localization functor for the category of fractions $\mathcal{M}[\mathcal{W}^{-1}]$, which means for any functor $F : \mathcal{M} \to \mathcal{D}$ taking weak equivalences to isomorphisms, there exists a unique functor $F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ such that $F_* \circ \gamma = F$.*

Proof. Assume $f, g : X \to Y$ in $\mathcal{M}$, if $f \simeq_r g$ or $f \simeq_l g$, then $F(f) = F(g)$. The proofs are the same. Hence we can just assume $h : f \simeq_r g$. $s : Y \to Y^I$ is a weak equivalence hence $F(s)$ an isomorphism. $p_i \circ s = \mathrm{id}_Y \Rightarrow F(p_i) = F(s)^{-1}$ and

$$f = p_0 \circ h, g = p_1 \circ h \Rightarrow F(f) = F(g) = F(s)^{-1} F(h)$$

On objects F_* is easily defined, for $\mathrm{Ho}(\mathcal{M})$ has the same objects as $\mathcal{M}$. Now suppose $[f] \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = [RQX, RQY]$ where $f : RQX \to RQY$.

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ & & & & \downarrow{\scriptstyle f} \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

where p_X, p_Y, j_X, j_Y are all weak equivalences. Thus

$$\begin{array}{ccccc} F(X) & \xleftarrow{\sim} & F(QX) & \xrightarrow{\sim} & F(RQX) \\ & & & & \downarrow{\scriptstyle F(f)} \\ F(Y) & \xleftarrow{\sim} & F(QY) & \xrightarrow{\sim} & F(RQY) \end{array}$$

We define

$$F_*([f]) = F(p_Y) \circ F(j_Y)^{-1} \circ F(f) \circ F(j_X) \circ F(p_X)^{-1}$$

If $[f] = [g]$, $f \simeq g$ then $F(f) = F(g)$. Hence F_* is well defined. It's obvious to see F_* is actually a functor. Now we need to prove $F_* \circ \gamma = F$. Given a map $f : X \to Y$, $\gamma(f) = [RQf] = [f_2]$.

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ Y & \xleftarrow[p_Y]{} & QY & \xrightarrow[j_Y]{} & RQY \end{array}$$

Take F in this diagram and we obtain:

$$\begin{array}{ccccc} F(X) & \xleftarrow{\sim} & F(QX) & \xrightarrow{\sim} & F(RQX) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(f_1)} & & \downarrow{\scriptstyle F(f_2)} \\ F(Y) & \xleftarrow{\sim} & F(QY) & \xrightarrow{\sim} & F(RQY) \end{array}$$

It's obvious to see $F_*\gamma(f) = F_*([f_2]) = F(f)$. Then we should prove F_* is unique.

Given $f : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$ that is $[f] \in [RQX, RQY]$. Since

$$RQ(QX) = RQX, \;\; RQ(RQX) = RQX$$

f can also represent an element of $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(QX, QY)$ and of $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(RQX, RQY)$. Consider the following diagram

$$\begin{array}{ccccc} QX & \xleftarrow{\mathrm{id}} & Q(QX) = QX & \xrightarrow{j_X} & RQ(RQX) = RQX \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xleftarrow[p_X]{} & QX & \xrightarrow[j_X]{} & RQX \end{array}$$

Hence $\gamma(p_X) = [\mathrm{id}_{RQX}]$.

$$\begin{array}{ccccc} QX & \xleftarrow{\mathrm{id}} & QX & \xrightarrow{j_X} & RQX \\ \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle \mathrm{id}} \\ RQX & \xleftarrow[\mathrm{id}]{} & RQX & \xrightarrow[\mathrm{id}]{} & RQX \end{array}$$

Then $\gamma(j_X) = [\mathrm{id}_{RQX}]$. Therefore we have the following commutative diagram in the category $\mathrm{Ho}(\mathcal{M})$:

$$\begin{array}{ccccc} X & \xleftarrow{\gamma(p_X)} & QX & \xrightarrow{\gamma(j_X)} & RQX \\ \downarrow{\scriptstyle [f]} & & \downarrow{\scriptstyle [f]} & & \downarrow{\scriptstyle [f]} \\ Y & \xleftarrow[\gamma(p_Y)]{} & QY & \xrightarrow[\gamma(j_Y)]{} & RQY \end{array}$$

$[f] \in [RQX, RQY]$, if $f : RQX \to RQY$ in $\mathcal{M}$, then $\gamma(f) = [f]$. $F = F_* \circ \gamma$ forces

$$F_*([f]) = F(p_Y) \circ F(j_Y)^{-1} \circ F(f) \circ F(j_X) \circ F(p_X)^{-1}$$

for $[f] : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$. Hence F_* is unique. □

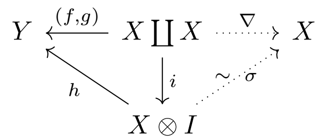
We can also consider the localization for $\mathcal{M}_c$ and $\mathcal{M}_f$ with respect to weak equivalences, and it's obvious to see $\mathcal{M}_c[\mathcal{W}^{-1}]$ (resp. $\mathcal{M}_c[\mathcal{W}^{-1}]$) is equivalent to the full subcategory of $\mathrm{Ho}(\mathcal{M})$ consisting of cofibrant objects (resp. fibrant objects), using the same method above. We can check the universal property of localization for subcategories $\mathrm{Ho}(\mathcal{M}_c)$ and $\mathrm{Ho}(\mathcal{M}_f)$ of $\mathrm{Ho}(\mathcal{M})$ directly. Moreover due to the existence of the factorization system in $\mathcal{M}$, $\mathrm{Ho}(\mathcal{M}_c)$ and $\mathrm{Ho}(\mathcal{M}_f)$ are all actually equivalent to $\mathrm{Ho}(\mathcal{W})$ [52, chapter 1, p1.13, theorem 1]. Finally we have the following commutative diagram:

$$\begin{array}{ccc} \pi\mathcal{M}_c & \xrightarrow{\bar{\gamma}_c} & \mathrm{Ho}(\mathcal{M}_c) \\ \uparrow & & \downarrow{\scriptstyle \cong} \\ \pi\mathcal{M}_{cf} & \xrightarrow[\sim]{\bar{\gamma}} & \mathrm{Ho}(\mathcal{M}) \\ \downarrow & & \uparrow{\scriptstyle \cong} \\ \pi\mathcal{M}_f & \xrightarrow{\bar{\gamma}_f} & \mathrm{Ho}(\mathcal{M}_f) \end{array}$$

Note that $\bar{\gamma}_c$ (resp. $\bar{\gamma}_f$) comes from the restriction of γ to $\mathcal{M}_c$ (resp. $\mathcal{M}$) and from the following lemma.

Lemma A.2.20. *Let $F : \mathcal{M} \to \mathcal{D}$ carry weak equivalences into isomorphisms. If $f \simeq_l g$ or $f \simeq_r$ then $F(f) = F(g)$.*

Proof. Since $f \simeq_l g$ and $f \simeq_r g$ are dual, we may assume $f \simeq_l g$. Let $h : X \otimes I \to Y$ be the left homotopy of f and g.



σ is a weak equivalence $\Rightarrow F(\sigma)$ is an isomorphism. Then for $i_0, i_1 : X \to X \otimes I$, $F(i_0) = F(i_1) = F(\sigma)^{-1}$. Hence

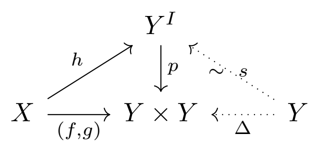
$$F(f) = F(h \circ i_1) = F(h) \circ F(\sigma)^{-1} = F(g)$$

□

Finally we want prove for the localization functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, if $\gamma(f)$ is an isomorphism, then f is a weak equivalence. Since homotopy equivalences are sent to isomorphisms via γ, this theorem will imply every homotopy equivalence is a weak equivalence. We give a proof here following [56] which is different from that in [52] and [23].

Lemma A.2.21. *$f, g : X \to Y$ are arbitrary two maps in $\mathcal{M}$. If $f \simeq_r g$ or $f \simeq_l g$, then f is a weak equivalence iff g is a weak equivalence.*

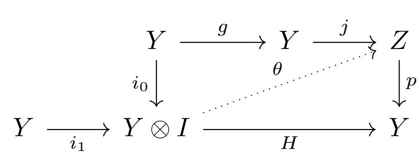
Proof. Proofs for the two conditions are the same. Hence we assume $f \simeq_r g$ and f is a weak equivalence.



In this diagram, p_i is weak equivalence for $i = 0, 1$. $f = p_0 \circ h \Rightarrow h$ is a weak equivalence. Hence $g = p_1 \circ h$ is a weak equivalence. □

Lemma A.2.22. *If X, Y are both cofibrant and fibrant, $f : X \to Y$ in $\mathcal{M}$ such that $\gamma(f)$ is an isomorphism, then f is weak equivalence.*

Proof. That $\gamma(f)$ is an isomorphism means f is homotopy equivalence between X and Y. We decompose f as $X \xrightarrow{j} Z \xrightarrow{p} Y$ where j is a trivial cofibration, p is a fibration and Z is both cofibrant and fibrant. We only need to prove p is a weak equivalence. Since f is a homotopy equivalence, there exists $g : Y \to X$ such that $gf \simeq \mathrm{id}_X, fg \simeq \mathrm{id}_Y$. If $H : fg \simeq id_Y$ is the left homotopy for a cylinder object $Y \otimes I$, then we have the following diagram:



where $pjg = fg = Hi_0$, i_0 is a trivial cofibration and p is a fibration.

We let $k = \theta \circ i_1 : Y \to Z$. $pk = p\theta i_1 = Hi_1 = \mathrm{id}_Y$. $\theta i_0 = jg$. Hence $\theta : jg \simeq_l k$. According to Whitehead's theorem, j is a homotopy equivalence. Then there is the homotopy inverse $q : Z \to X$ such that $qj \simeq \mathrm{id}_X, jq \simeq \mathrm{id}_Z$.

From

$$jq \simeq \mathrm{id}_Z,\ jg \simeq k,\ gf \simeq \mathrm{id}_X$$

we conclude

$$kp \simeq kpjq = kfq \simeq jgfq \simeq jq \simeq \mathrm{id}_Z$$

Hence kp is a weak equivalence.

$$\begin{array}{ccccc} Z & \xrightarrow{\mathrm{id}_Z} & Z & \xrightarrow{\mathrm{id}_Z} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle kp} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[k]{} & Z & \xrightarrow[p]{} & Y \end{array}$$

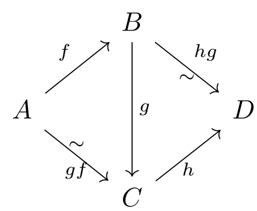
p is the retraction of kp. Hence p is a weak equivalence. ☐

Corollary A.2.23. *For any $f : X \to Y$ in $\mathcal{M}$, if $\gamma(f)$ is an isomorphism, then f is a weak equivalence.*

Proof. That $\gamma(f)$ is an isomorphism means f_2 is a homotopy equivalence. According to the Lemma A.2.22, f_2 is a weak equivalence. Hence f is also a weak equivalence. ☐

This corollary tells us that the class $\mathcal{W}$ of weak equivalences coincides with the class of morphisms which are inverted by the functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. This fact tells us that in a model category weak equivalences satisfy a property called *two out of six*, which is similar to the property called two out of three appearing in the definition of model categories.

Corollary A.2.24. *Given a commutative diagram*

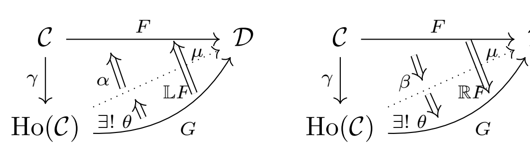


If gf and hg are weak equivalences, then so are $f,\ g,\ h,\ hgf$.

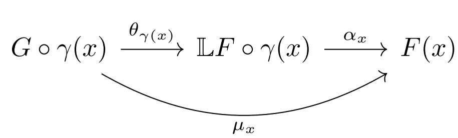
Proof. Take the functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ to this diagram. $\gamma(gf), \gamma(hg)$ are isomorphisms. Then $\gamma(g)$ has a right inverse $\gamma(f) \circ \gamma(gf)^{-1}$ and a left inverse $\gamma(hg)^{-1} \circ \gamma(h)$. Hence $\gamma(g)$ is an isomorphism. The Corollary A.2.23 tells us g will be a weak equivalence. Then f, h, hgf are all weak equivalences. ☐

A.3 Derived Functors

Definition A.3.1. Let $\mathcal{C}$ be a model category with the localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ and a functor $F : \mathcal{C} \to \mathcal{D}$. The *left derived functor* of F is $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ together with a natual transformation $\alpha : \mathbb{L}F \circ \gamma \Rightarrow F$ which turns $\mathbb{L}F$ into the *right Kan extension* along γ. Dually the *right derived functor* is the *left Kan extension* with $\beta : F \Rightarrow \mathbb{R}F \circ \gamma$.



For the right Kan extension we mean given any other functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ and a transformation $\mu : G \circ \gamma \Rightarrow F$, there exists a unique transformation $\theta : G \Rightarrow \mathbb{L}F$ such that $\alpha \circ \theta = \mu$.



The left Kan extension is defined dually.

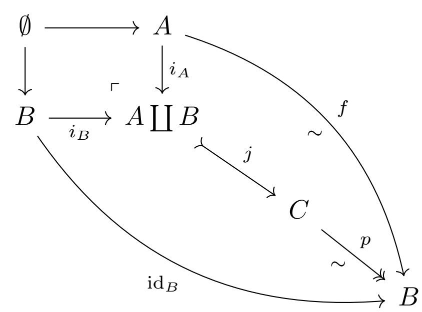
Note that from the definition of derived functors, we see they are independent from model structures on $\mathcal{C}$ and are only relative to its homotopy theory.

Lemma A.3.2 (Ken Brown)**.** *Suppose $F : \mathcal{C} \to \mathcal{M}$ is a functor from a mdoel category $\mathcal{C}$ to a category $\mathcal{N}$ with a class of weak equivalences satisfying the two-of-three property.*

(1). If F sends trivial cofibrations between cofibrant objects to weak equivalences then it sends weeak equivalences between cofibrant objects to weak equivalences.

(2). If F sends trivial fibrations between fibrant objects to weak equivalences then it sends weeak equivalences between fibrant objects to weak equivalences.

Proof. Since the two statements are dual, we just prove the first one. Suppose $f : A \xrightarrow{\sim} B$ is a weak equivalence between cofibrant objects.



In the diagram above there is a unique map $(f, \mathrm{id}_B) : A \coprod B \to B$. We factor this map as $p \cdot j$ where j is a cofibration and p is a tricial fibration. Note that C is cofibrant. By the two-of-three property $j \cdot i_A$ and $j \cdot i_B$ are trivial cofibrations. $F(\mathrm{id}_B) = F(p) \cdot F(j \cdot i_B)$. Therefore $F(p)$ is a weak equivalence in $\mathcal{N}$. Then $F(f) = F(p) \cdot F(j \cdot i_A)$ is a weak equivalence. □

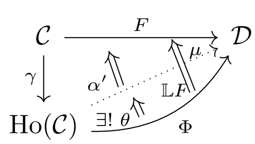
Proposition A.3.3. *For any functor $F : \mathcal{C} \to \mathcal{D}$ sending trivial cofibrations between cofibrant objects to isomorphisms where $\mathcal{C}$ is a model category, the left derived functor $(\mathbb{L}F, \alpha)$ exists which is absolute in the sense that given any other functor $G : \mathcal{D} \to \mathcal{E}$, $(G \cdot \mathbb{L}F, G\alpha)$ is the left derived functor of $G \circ F$.*

There is a dual version of this proposition for right derived functors.

Proof. By Ken Brown's lemma, F sends weak equivalences between cofibrant objects to isomorphisms. Therefore the restriction of F to $\mathcal{C}_c$ factors as

$$\mathcal{C}_c \xrightarrow{\gamma_c} \mathrm{Ho}(\mathcal{C}_c) \xrightarrow{F_c} \mathcal{D}$$

But $\mathrm{Ho}(\mathcal{C}_c) \cong \mathrm{Ho}(\mathcal{C})$. Then $\mathbb{L}F$ is defined to be $\mathrm{Ho}(\mathcal{C}) \underset{\sim}{\xrightarrow{Q}} \mathrm{Ho}(\mathcal{C}_c) \xrightarrow{F_c} \mathcal{D}$ where Q is obtained by choosing a cofibrant replacement QX for all $X \in \mathrm{Ob}(\mathcal{C})$ with a trivial fibration $\alpha_X : QX \to X$. Note that for $f : X \to Y$ in $\mathrm{Ho}(\mathcal{C})$, $Q(f) = \alpha_Y^{-1} \cdot f \cdot \alpha_X$.



We set $\alpha'_X = F(\alpha_X) : F(QX) \to F(X)$ and we need to prove α' is actually a natural transformation first. For any map $f : X \to Y$ there is a diagram

$$\begin{array}{ccc} F(QX) & \xrightarrow{F(\alpha_X)} & F(X) \\ {\scriptstyle F_c(\alpha_Y^{-1} f \alpha_X)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(QY) & \xrightarrow[F(\alpha_Y)]{} & F(Y) \end{array}$$

But since QX is cofibrant and α_Y is a trivial fibration, there will exist a morphism $g : QX \to QY$ satisfying $\alpha_Y g = f\alpha_X$. Therefore $\alpha_Y^{-1} f\alpha_X$ in $\mathrm{Ho}(\mathcal{C}_c)$ admits a preimage g in $\mathcal{C}_c$. Therefore $F_c(\alpha_Y^{-1} f\alpha_X) = F(g)$ and the diagram above is commutative.

Next given any functor $\Phi : \mathrm{Ho}(\mathcal{C} \to \mathcal{D})$ and a natural transformation $\mu : \Phi \Rightarrow F \circ \gamma$ we need to there is a unique transformation $\theta : \Phi \Rightarrow F_c \circ Q$ satisfying $\mu = \alpha' \circ \theta$.

$$\Phi(X) \overset{\theta_X}{\dashrightarrow} F_c(QX) = F(QX) \xrightarrow{F(\alpha_X)} F(X), \qquad \mu_X : \Phi(X) \to F(X)$$

If X is cofibrant, $F(\alpha_X)$ will be an isomorphism. Then $\theta_X = F(\alpha_X)^{-1} \cdot \mu_X$ is unique. And since $\mathrm{Ho}(\mathcal{C}_c)$ is equivalent to $\mathrm{Ho}(\mathcal{C})$, θ can be extended to be a unique natural transformation on $\mathrm{Ho}(\mathcal{C})$.

Finally for any other functor $G : \mathcal{D} \to \mathcal{E}$, $\mathbb{L}(G \circ F) = (G \circ F)_c \circ Q = G \circ F_c \circ Q = G \circ \mathbb{L}F$. □

Remark A.3.4. If we just deal with homotopical categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *left deformable* if there exists a *left deformation* i.e. an endofunctor $Q : \mathcal{C} \to \mathcal{C}$ with a natural weak equivalence $Q \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}$ such that on $Q(\mathcal{C})$, F sends weak equivalences to weak equivalences. Then we will have the left derived functor $\mathbb{L}F = F \circ Q$ [56, Theorem 4.1.7].

Definition A.3.5. Suppose $\mathcal{C}$ and $\mathcal{D}$ are two model categories. A *Quillen adjunction* is a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ such that F preserves cofibrations and G preserves fibrations.

Remark A.3.6. According to Corollary A.2.2 and the adjointness, the definition above is also equivalent to saying F preserves cofibrations and trivial cofibrations or G preserves fibrations and trivial fibrations.

Corollary A.3.7. *For any Quillen adjuncion $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ the left derived functor $\mathbb{L}F$ and the right derived functor $\mathbb{R}G$ exist. Moreover they form a pair of adjoint functors*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$$

Proof. The existence is a direct consequence of the proposition above. Hence we only need to prove the second part.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbb{L}FX, Y) &= \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FQX, Y) \\
&= \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FQX, PY), \ PY \text{ is the fibrant replacement} \\
&= \mathrm{Hom}_{\mathcal{D})}(FQX, PY)/\sim, \ \text{Proposition A.2.18} \\
&= \mathrm{Hom}_{\mathcal{C}}(QX, GPY)/\sim, \ \text{Corollary A.2.12 and adjointness} \\
&= \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, \mathbb{R}GY)
\end{aligned}$$

□

A classical theorem for adjoint functors says the left (resp. right) adjoint functor preserves colimits (resp. limits). There is a similar theorem for Quillen pairs and homotopy colimits (limits)[22].

Theorem A.3.8. *For any Quillen adjuncion $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ the left (resp. right) derived functor $\mathbb{L}F$ (resp. $\mathbb{R}G$) preserves homotopy colimits (resp. limits).*

[22] Details of homotopy colimits and limits containing the definition can be found in the next section.

Proof. We only need to prove $\mathbb{L}F$ preserves homotopy colimits. For simplicity we suppose model categories $\mathcal{C}$ and $\mathcal{D}$ are *combinatorial* and theorefore they admit *injective* and *projective* model structures on functor categories[23]. [24]

The Quillen pair (F, G) can induce a new Quillen pair $F_* : \mathcal{C}_{proj} \rightleftarrows \mathcal{D}_{proj} : G_*$ where for any I-indexed diagram $X : I \to \mathcal{C}$, $F_*(X) = F \circ X$. Let $\mathrm{cons} : \mathcal{C}$ (resp. $\mathcal{D}$) $\to \mathcal{C}^I$ (resp. $\mathcal{D}^I$) be the constant functor. Then we have

$$\begin{array}{ccc} \mathcal{D}^I & \xrightarrow{G_*} & \mathcal{C}^I \\ {\scriptstyle \mathrm{cons}}\uparrow & & \uparrow{\scriptstyle \mathrm{cons}} \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

This means $G_* \cdot \mathrm{cons} = \mathrm{cons} \cdot G$, which implies $\mathrm{Ho}(G_*) \cdot \mathbb{R}\mathrm{cons} = \mathrm{Ho}(\mathrm{cons}) \cdot \mathbb{R}G$. For the constant functor, $\mathbb{R}\mathrm{cons} = \mathrm{cons} \cdot P$ is just the fibrant replacement. For any object Y of $\mathcal{D}$, since PY is fibrant and G_* preserves fibrant objects, $\mathrm{Ho}(G_*) \cdot \mathbb{R}\mathrm{cons}(Y)$ and $\mathbb{R}G_* \cdot \mathbb{R}\mathrm{cons}(Y)$ are canonically isomorphic on homotopy categories. Similarly $\mathrm{Ho}(\mathrm{cons}) \cdot \mathbb{R}G$ and $\mathbb{R}\mathrm{cons} \cdot \mathbb{R}G$ are canonically isomorphic as well. Therefore $\mathbb{R}G_* \cdot \mathbb{R}\mathrm{cons}$ and $\mathbb{R}\mathrm{cons} \cdot \mathbb{R}G$ are canonically isomorphic. Finally since $\mathbb{R}\mathrm{cons}$ is right adjoint to $\mathrm{hocolim}$. passing to the left adjoints, $\mathbb{L}F \cdot \mathrm{hocolim}$ and $\mathrm{hocolim} \cdot \mathbb{L}F$ are canonically isomorphic. □

A.4 Homotopy Limits and Colimits

Definition A.4.1. Suppose I is a small category and $\mathcal{C}$ is a model category. Weak equivalences in $\mathcal{C}^I$ are objectwise weak equivalences. Then the *homotopy colomit* functor, if exists, is the left derived functor

$$\mathrm{hocolim} := \mathbb{L}\mathrm{colim} : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$$

and the *homotopy limit* functor, if exists, is the right derived functor

$$\mathrm{holim} := \mathbb{R}\mathrm{lim} : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$$

Although the existence of derived functors are independent from model structures, in order to use Proposition A.3.3 to compute homotopy limits and colimits we need to know some specific model structure on $\mathcal{C}^I$. Since weak equivalences on $\mathcal{C}^I$ are objectwise, it's natural to ask whether there exists some model structure on $\mathcal{C}^I$ making cofibrations or fibrations objectwise.

Definition A.4.2. (1). The *projective model structure* on $\mathcal{C}^I$ has weak equivalences and fibrations defined objectwise.

(2). The *injective model structure* on $\mathcal{C}^I$ has weak equivalences and cofibrations defined objectwise.

Theorem A.4.3. *Fro any cocomplete cofibrantly generated model category $\mathcal{C}$ which permits the small object argument, and any small category I, the functor category $\mathcal{C}^I$ admits a projective model structure.*

To prove this theorem we need some preparations. Let $\mathcal{C}$ be a cocomplete category. If X is an object of $\mathcal{C}$ and S is a set, then we define $X \otimes S := \coprod_{s \in S} X$. If $S : I \to \mathbf{Set}$ is a functor, then we can define a functor from I to $\mathcal{C}$ for any object X of $\mathcal{C}$ as follows

$$X \otimes S : I \to \mathcal{C},\ \alpha \mapsto X \otimes S_\alpha = \coprod_{s \in S_\alpha} X$$

[23]See Remark A.4.5

[24]I think there is also a categorical proof using the universal property of Kan extensions for any functorial model category. But the proof we discuss here is enough.

For any map $f : \alpha \to \beta$ in I, we have $S_f : S_\alpha \to S_\beta$

$$\begin{array}{ccc} X \otimes S_\alpha = \coprod_{S_\alpha} X & \overset{\exists!}{\dashrightarrow} & X \otimes S_\beta = \coprod_{S_\beta} X \\ \nwarrow & & \nearrow \\ X_{s \in S_\alpha} & \xrightarrow[\mathrm{id}_X]{} & X_{S_f(s) \in S_\beta} \end{array}$$

This induces a map $X \otimes S_\alpha \to X \otimes S_\beta$. Therefore $X \otimes S$ is actually a functor.

If $S = \mathrm{Hom}_I(\alpha, -)$, then $X \otimes S_\beta = \coprod_{\mathrm{Hom}_I(\alpha,\beta)} X$. And

$$- \otimes \mathrm{Hom}_I(\alpha, -) : \mathcal{C} \rightleftarrows \mathcal{C}^I : ev_\alpha$$

is an adjoint pair where $ev_\alpha(F) = F(\alpha)$. To prove

$$\mathrm{Hom}_{\mathcal{C}^I}(X \otimes \mathrm{Hom}_I(\alpha, -), F) \cong \mathrm{Hom}_{\mathcal{C}}(X, F(\alpha))$$

we see that a natural transformstion $X \otimes \mathrm{Hom}_I(\alpha, -) \to F$ consists of maps $X \otimes \mathrm{Hom}_I(\alpha, \beta) \to F(\beta)$ for all $\beta \in \mathrm{Ob}(I)$ such that for any map $f : \beta \to \gamma$ in I, the following diagram is commutative

$$\begin{array}{ccc} \coprod_{\mathrm{Hom}_I(\alpha,\beta)} X & \longrightarrow & F(\beta) \\ \downarrow & & \downarrow F_f \\ \coprod_{\mathrm{Hom}_I(\alpha,\gamma)} X & \longrightarrow & F(\gamma) \end{array}$$

which is equivalent to saying for all $\mu : \alpha \to \beta$

$$X_\mu \to F(\beta) \xrightarrow{F_f} F(\gamma) = X_{f \cdot \mu} \to F_\gamma$$

But all of these are determined by the map $X_{\mathrm{id}_\alpha} \to F(\alpha)$ since for $\mu : \alpha \to \beta$

$$X_\mu \to F(\beta) = X_{\mathrm{id}_\alpha} \to F(\alpha) \xrightarrow{F_\mu} F(\beta)$$

Therefore the adjunctiojn above is proved.

Proof of Theorem A.4.3. Suppose K and J are the generating classes for cofibrations and trivial cofibrations respectively in $\mathcal{C}$. Using the adjunction

$$- \otimes \mathrm{Hom}_I(\alpha, -) : \mathcal{C} \rightleftarrows \mathcal{C}^I : ev_\alpha$$

given the cardinal κ and an indexed diagram $F : \kappa \to \mathcal{C}^I$, then for any map $k : X \to Y$ in K we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^I}(X \otimes \mathrm{Hom}_I(\alpha, -), \mathrm{colim}_\kappa F_u) &\cong \mathrm{Hom}_{\mathcal{C}}(X, ev_\alpha(\mathrm{colim}_\kappa F_u)),\ u \in \kappa \\ &\cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{colim}_\kappa ev_\alpha F_u) \\ &\cong \mathrm{colim}_\kappa \mathrm{Hom}_{\mathcal{C}}(X, ev_\alpha F_u) \\ &\cong \mathrm{colim}_\kappa \mathrm{Hom}_{\mathcal{C}^I}(X \otimes \mathrm{Hom}_I(\alpha, -), F_u) \end{aligned}$$

Therefore the small object argument is valid for maps $- \otimes \mathrm{Hom}_I(\alpha, -)(K)$ and $- \otimes \mathrm{Hom}_I(\alpha, -)(J)$.

For the map $k : X \to Y$, the map $k \otimes \mathrm{Hom}_I(\alpha, -)$ actaully consists of maps

$$\coprod_{\mathrm{Hom}_I(\alpha,\beta)} k : \coprod_{\mathrm{Hom}_I(\alpha,\beta)} X \to \coprod_{\mathrm{Hom}_I(\alpha,\beta)} Y$$

for all $\beta \in \mathrm{Ob}(I)$. Since here α is an arbitrary object in I, the generating cofibrations and trivial cofibrations in $\mathcal{C}^I$ should be

$$K_I = \cup_\alpha - \otimes \mathrm{Hom}_I(\alpha, -)(K), \text{ and } J_I = \cup_\alpha - \otimes \mathrm{Hom}_I(\alpha, -)(J)$$

respectively.

Since $- \otimes \mathrm{Hom}_I(\alpha, -) \dashv ev_\alpha$, $f \in r(- \otimes \mathrm{Hom}_I(\alpha, -)(K))$ if and only if $ev_\alpha f \in r(K)$. Hence $(- \otimes \mathrm{Hom}_I(\alpha, -)(K)) = ev_\alpha^{-1}(r(K))$. Then

$$r(K_I) = \cap_\alpha r(- \otimes \mathrm{Hom}_I(\alpha, -)(K)) = \cap_\alpha ev_\alpha^{-1}(r(K))$$

Given a map f in $\mathcal{C}^I$, $f \in r(K_I)$ iff for all α in I, $ev_\alpha f = f_\alpha \in r(K)$ which means f_α is a trivial fibration in $\mathcal{C}$. Therefore $r(K_I)$ and $r(J_I)$ consist of objectwise trivial fibrations and fibrations respectively.

Let W be the class of objectwise weak equivalences in $\mathcal{C}^I$. Then we only need to show $l \circ r(J_I) = l \circ r(K_I) \cap W$.

Suppose $\mu : F \to G$ belongs to $l \circ r(J_I)$, then μ has the LLP wrt all objectwise fibrations.

$$F \xrightarrow[\in\, l\circ r(J_I)]{\mu_1} \bullet \xrightarrow[\in\, r(J_I)]{\mu_2} G, \qquad \mu = \mu_2\mu_1$$

By Lemma A.1.2, μ will be the retract of $\mu_1 \in l \circ r(J_I)$. But note that for $j : X \to Y \in J$, it's a trivial cofibration in $\mathcal{C}$. And since $Cof \cap W_\mathcal{C}$ is saturated in $\mathcal{C}$, in the process of small object argument (see Theorem A.1.11) μ_1 will finally be an objectwise trivial cofibration. Thus its retract μ will be an objectwise weak equivalence.

Conversely if $\mu \in l\circ r(K_I) \cap W$. Factor μ as the same as above. Since μ_1 is an objectwise trivial cofibration and μ is an objectwise weak equivalence, μ_2 will be an objectwise trivial fibration hence in $r(K_I)$. Then μ will be the retract of μ_1 in $l \circ r(J_I)$. □

There is an abstract summary of the method we used in the proof to promote a model structure on a category to another category via an adjoint pair [24, Theorem 3.6].

Theorem A.4.4. *Let* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *be an adjoint pair and suppose* $\mathcal{C}$ *is a cofibrantly generated model category. Let* I *and* J *be the sets of generating cofibrations and trivial cofibrations respectively. We define a map* f *in* $\mathcal{D}$ *is a weak equivalence or a fibration if* $G(f)$ *in* $\mathcal{C}$ *is a weak equivalence or a fibration. Then if the following conditions satisfied,*

(1). the right adjoint G *commutes with sequential colimits;*

(2). every map in $\mathcal{D}$ *having the LLP wrt all fibrations is a weak equivalence;*

then $\mathcal{D}$ *will become a cofibrantly generated model category such that* $F(I)$ *and* $F(J)$ *generate its cofibrations and trivial cofibrations respectively.*

Remark A.4.5. Actually if $\mathcal{C}$ is a *combinatorial mdoel category* (locally presentable[25] and cofibrantly generated) then $\mathcal{C}^I$ admits both projective and injective model structures [38, Proposition A.2.8.2]. And actually nearly all of model categories we meet are combinatorial or Quillen equivalent to a combinatorial one.

Theorem A.4.6. *Let* $\mathcal{C}$ *be a model category and* I *be a small category.*

(1). Whenever the projective model structure on $\mathcal{C}^I$ *exists, then the homotopy colimit* $\mathbb{L}\mathrm{colim} : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$ *exists and can be computed as the colimit of a projective cofibrant replacement of the original diagram.*

(2). Whenever the injective model structure on $\mathcal{C}^I$ *exists, then the homotopy limit* $\mathbb{R}\lim : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$ *exists and can be computed as the limit of an injective fibrant replacement of the original diagram.*

[25] There are many equivalent definitions of the property being *locally presentable*. An intuitive way is to use the concept of *accessible category*. And another way is to define it as a full subcategory of a category of presheaves satisfying some adjointness condition which is just C. Rezk done in [54]. Details of these can be found in [4].

Proof. The two satements are dual and hence we only prove the first. Consider the adjoint pair $\mathrm{colim} : \mathcal{C}^I \rightleftarrows \mathcal{C} : \mathrm{cons}$ where cons is the constant functor. Then cons sends fibrations to fibrations and trivial fibrations to trivial fibrations. Therefore they form a Quillen pair and then the Corollary A.3.7 can be applied. □

A.4.1 Homotopy Pullbacks and Pushouts

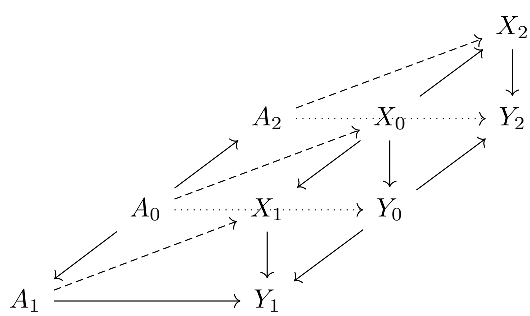
In the following we try to compute homotopy pullbacks and homotopy pushouts which means $I := 1 \to 0 \leftarrow 2$ and $I := 1 \leftarrow 0 \to 2$ respectively. Since they are dual, we give the example of homotopy pushouts.

Proposition A.4.7. *The pushout diagram $A_1 \leftarrowtail A_0 \rightarrowtail A_2$ consisting of a pair of cofibrations between cofibrant objects is projectively cofibrant.*

Proof. Given any trivial fibration $X_\bullet \to Y_\bullet$ in $\mathcal{C}^I$ and any lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ A_\bullet & \longrightarrow & Y_\bullet \end{array}$$

we expend this lifting problem first.



At first since A_0 is cofibrant and $X_0 \to Y_0$ is a trivial fibration, we can obtain a map $A_0 \to X_0$. Then we solve the lifting problem $(A_0 \to A_1, X_1 \to Y_1)$ where $A_0 \to X_1$ is the composition of $X_0 \to X_1$ and $A_0 \to X_0$. The map $A_2 \to X_2$ is obtained similarly. □

Remark A.4.8. For any diagram $B_\bullet$ in $\mathcal{C}^I$ which is equipped with the projective model structure, we can construct its cofibrant replacement as follows.

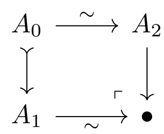
$$\begin{array}{ccccc} A_1 & \leftarrowtail & A_0 & \rightarrowtail & A_2 \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ B_1 & \longleftarrow & B_0 & \longrightarrow & B_2 \end{array}$$

First we replace B_0 by its cofibrant replacement A_0. Then we factor $A_0 \to B_1$ as a composition of a cofibration and a trivial fibration. $A_0 \to B_2$ is factored similarly. Then by the proposition above $A_\bullet$ is projectively cofibrant.

For some special model categories, the computation of homotopy pullbacks and pushouts can be much simpler.

Definition A.4.9. A model category is called

(1). *left proper* if weak equivalences are preserved by pushouts along cofibrations.



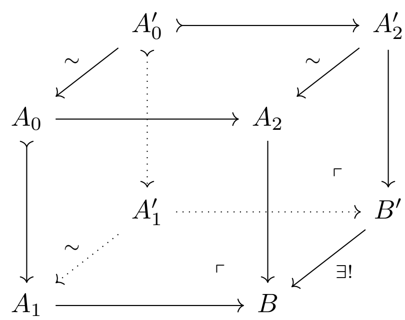
(2). *righ proper* if weak equivalences are preserved by pullbacks along fibrations

(3). *proper* if it's both left proper and right proper.

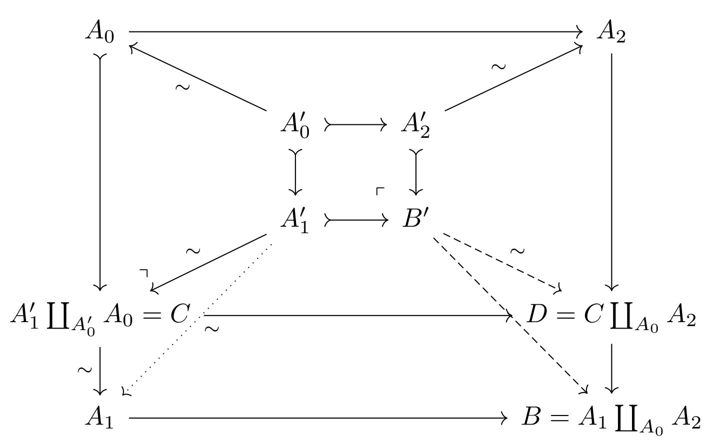
Theorem A.4.10. *(1). In a left proper model category, ordinary pushout along cofibrations are homotopy pushouts.*

(2). In a right proper model category, ordinary pullbacks along fibrations are homotopy pullbacks.

Proof. We just prove the first statement since they are dual. Given a pushout diagram $A_1 \leftarrowtail A_0 \to A_2$ we need to show $B = A_1 \coprod_{A_0} A_2$ is the homotopy pushout.



At first we replace the diagram $A_1 \leftarrowtail A_0 \to A_2$ by its cofibrant replacement $A_1' \leftarrowtail A_0' \rightarrowtail A_2'$ according to Remark A.4.8. Let $B' = A_1' \coprod_{A_0'} A_2'$. Then there will exist a unique map $B' \to B$. We will prove this map is a weak equivalence hence an isomorphism in $\mathrm{Ho}(\mathcal{C})$.



Let $C = A_1' \coprod_{A_0'} A_0$. Since $\mathcal{C}$ is left proper, $A_1' \to C$ is a weak equivalence. Note that $A_1' \to A_1$ is a weak equivalence. Hence the unique map $C \to A_1$ is a weak equivalence.

Next we prove $D = C \coprod_{A_0} A_2 \cong B' \coprod_{A_2'} A_2$ and then this will imply the map $B' \to D$ is a weak equivalence by the left properness. Given a pair of maps $(B' \to X, A_2 \to X)$ which coincide on A_2'. $(A_1' \to B' \to X, A_0 \to A_2 \to X)$ will induce a unique map $C \to X$. $(C \to X, A_2 \to X)$ gives a unique map $D \to X$. This proves D is the pushout of $B' \leftarrowtail A_2' \xrightarrow{\sim} A_2$.

Finally to prove $B' \to B$ is a weak equivalence we only need to prove $D \to B$ is a weak equivalence.

$$\begin{array}{ccccc} A_0 & \rightarrowtail & X & \overset{\sim}{\twoheadrightarrow} & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ C & \rightarrowtail & Y & \xrightarrow{\sim} & D \\ {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow & & \downarrow \\ A_1 & \rightarrowtail & Z & \longrightarrow & B \end{array}$$

We factor $A_0 \to A_2$ as a composition of a cofibration $A_0 \to X$ and a trivial fibration $X \to A_2$. Let $Y = C \coprod_{A_0} X$. Then D will be the pushout $Y \coprod_X A_2$ by pasting theorem for pushouts. Next let $Z = A_1 \coprod_C Y$. Then B will be the pushout $Z \coprod_Y D$. Since $\mathcal{C}$ is left proper, along $X \rightarrowtail Y$, $Y \to D$ is a weak equivalence, and along $C \rightarrowtail Y$, $Y \to Z$ is a weak equivalence. Since $A_0 \to A_1$ is a cofibration by assumption, $X \to Z$ will also be a cofibration along which $Z \to B$ will be a weak equivalence. Then $D \to B$ is hence a weak equivalence. ☐

Example A.4.11. [14, Proposition 2.3.27] implies a model category with all objects cofibrant is left proper. Therefore $s\mathbf{Set}$ is left proper. Actually it's right proper as well.

A.4.2 Reedy Categories

Until now we have only considered projective and injective model structures on $\mathcal{C}^I$. But for some special small categories I, there will exist another model structure on $\mathcal{C}^I$ with objectwise weak equivalences.

Definition A.4.12. A *Reedy structure* on a small category $\mathcal{R}$ consists of a degree map $\deg : \mathrm{Ob}(\mathcal{R}) \to \mathbb{Z}_{\geq 0}$ together with a pair of subcategories $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$ of degree-increasing and decreasing arrows respectively such that

(1). for any non-identity morphism in $\overrightarrow{\mathcal{R}}$, the degree of the domain is strictly less than the degree of the codomain and dually for any non-identity map in $\overleftarrow{\mathcal{R}}$ the degree of the domain is strictly greater than the degree of the codomain

(2). any morphism f in $\mathcal{R}$ can be uniquely factored as

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad f \quad} & \bullet \\ & {\scriptstyle \overleftarrow{f} \in \overleftarrow{\mathcal{R}}} \searrow \quad \nearrow {\scriptstyle \overrightarrow{f} \in \overrightarrow{\mathcal{R}}} & \\ & \bullet & \end{array}$$

A category $\mathcal{R}$ with Reedy structure is called the *Reedy category*.

Example A.4.13. Δ is a Reedy category with $\overrightarrow{\Delta}$ and $\overleftarrow{\Delta}$ consisting of injective maps and surjective maps respectively.

Fact A.4.14. (1). If $\mathcal{R}$ is a Reedy category, then $\mathcal{R}^{op}$ is also a Reedy category with $\overrightarrow{\mathcal{R}^{op}} = \overleftarrow{\mathcal{R}}^{op}$ and $\overleftarrow{\mathcal{R}^{op}} = \overrightarrow{\mathcal{R}}^{op}$.

(2). If $\mathcal{R}$ and $\mathcal{R}'$ are two Reedy categories, then $\mathcal{R} \times \mathcal{R}'$ will also be a Reedy category with the degree map being the plus of the two.

Therefore Δ^{op} is Reedy as well. For a category $\mathcal{C}$, *simplicial objects* and *cosimplicial* objects on it are functors $\Delta^{op} \to \mathcal{C}$ and $\Delta \to \mathcal{C}$ respectively.

Theorem A.4.15. [26] *For any Reedy category $\mathcal{R}$ and model category $\mathcal{C}$.*

(1). the category $\mathcal{C}^{\overrightarrow{\mathcal{R}}}$ admits a projective model structure

(2). the category $\mathcal{C}^{\overleftarrow{\mathcal{R}}}$ admits an injective model structure

(3). the category $\mathcal{C}^{\mathcal{R}}$ admits a model structure called the Reedy model structure such taht a map in it is a (Reedy) weak equivalence, a (Reedy) cofibration or a (Reedy) fibration iff its restriction to both $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$ is so in the sense above.

We can also describe the Reedy model structure more explicitly.

Definition A.4.16. Let $\mathcal{R}$ be a Reedy category and α be an object of it.

(1). The *latching category* $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ of $\mathcal{R}$ at α is the full subcategory of $\overrightarrow{\mathcal{R}} \downarrow \alpha$[27] containing all objects except the identity map id_α.

(2). The *matching category* $\partial(\alpha \downarrow \overleftarrow{\mathcal{R}})$ of $\mathcal{R}$ at α is the latching category $\partial(\overrightarrow{\mathcal{R}^{op}} \downarrow \alpha)$ of $\mathcal{R}^{op}$ at α.

Definition A.4.17. Let $\mathcal{C}$ be a model category and $X : \mathcal{R} \to \mathcal{C}$ be a functor from a Reedy category $\mathcal{R}$ to $\mathcal{C}$. Here we also use X to denote the induced functor on $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ sending $\beta \to \alpha$ to $X(\beta)$. Then

(1). the *latching object* of X at α is $L_\alpha := \operatorname{colim}_{\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)} X$ and the *latching map* of X at α is the natural map $L_\alpha X \to X(\alpha)$

(2). the *matching object* of X at α is $M_\alpha := \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{R}})} X$ and the *matching map* of X at α is the natural map $X(\alpha) \to M_\alpha X$.

If $f : X \to Y$ is a morphism in $\mathcal{C}^{\mathcal{R}}$, then

(1). the *relative latching map* of f at α is the natural map $X(\alpha) \coprod_{L_\alpha X} L_\alpha Y \to Y(\alpha)$

(2). the *relative matching map* of f at α is the natural map $X(\alpha) \to Y(\alpha) \times_{M_\alpha Y} M_\alpha X$

Theorem A.4.18. [28] *Let $\mathcal{R}$ be a Reedy category and $\mathcal{C}$ be a model category. For a map $f : X \to Y$ in $\mathcal{C}^{\mathcal{R}}$*

(1). it's a Reedy weak equivalence if it's an objectwise weak equivalence

(2). it's a Reedy (trivial) cofibration if for every α in $\mathcal{R}$, the relative latching map $X(\alpha) \coprod_{L_\alpha X} L_\alpha Y \to Y(\alpha)$ is a (trivial) cofibration in $\mathcal{C}$

(3). it's a Reedy (trivial) fibration if for every α in $\mathcal{R}$, the relative matching map $X(\alpha) \to Y(\alpha) \times_{M_\alpha Y} M_\alpha X$ is a (trivial) fibration in $\mathcal{C}$.

Moreover if $\mathcal{C}$ is a simplicial model category[29], then with the Reedy model structure $\mathcal{C}^{\mathcal{R}}$ will also be a simplicial model category.

Definition A.4.19. Let $\mathcal{C}$ be a model category and X be an object in $\mathcal{C}$. Viewing X as the constant functor in $\mathcal{C}^{\Delta}$ and $\mathcal{C}^{\Delta^{op}}$, then

(1). the *cosimplicial resolution* $\widetilde{X}$ is a cofibrant replacement of X in the Reedy model category $\mathcal{C}^{\Delta}$

[26] [17, Proposition 22.3]

[27] Objects in this category are morphisms from an object in $\overrightarrow{\mathcal{R}}$ to α

[28] [17, 22.6] or [28, Theorem 15.3.4]

[29] For the theory of simplicial model categories, see the next section.

(2). the *simplicial resolution* $\widehat{X}$ is a fibrant replacement of X in the Reedy model category $\mathcal{C}^{\Delta^{op}}$.

Remark A.4.20. The Reedy model structure and the projective model structure on $\mathcal{C}^{\mathcal{R}}$ are not equal generally but they are always Quillen equivalent [28, section 15.6].

Lemma A.4.21. [30] *Let $\mathcal{R}$ be a Reedy category and $\mathcal{C}$ be a model category. Suppose $f : X \to Y$ is a map between $\mathcal{R}$-indexed diagrams, α is an object of $\mathcal{R}$ and S is a class of maps in $\mathcal{C}$.*

(1). If for every object β of $\mathcal{R}$ whose degree is less than that of α the relative latching map $X(\beta) \coprod_{L_\beta X} L_\beta Y \to Y(\beta)$ has the LLP wrt all maps in S, then the induced map $L_\alpha X \to L_\alpha Y$ of latching objects will also have the LLP wrt all maps in S.

(2). Dually If for every object β of $\mathcal{R}$ whose degree is less than that of α the relative matching map $X(\alpha) \to Y(\beta) \times_{M_\beta Y} M_\beta X$ has the RLP wrt all maps in S, then the induced map $L_\alpha X \to L_\alpha Y$ of latching objects will also have the RLP wrt all maps in S.

Proof. We only prove the first statement. At first we decompose $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ as follows

$$F^0\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha) \subseteq F^1\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha) \subseteq \cdots \subseteq F^{\deg(\alpha)-1}\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha) = \partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$$

where $F^k\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ is the full subcategory of $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ consisting of objects $\beta \to \alpha$ in$\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ such that $\deg(\beta) \leq k$. Then in $F^0\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ only identity morphisms id_β with $\deg(\beta) = 0$ are allowed and for such β, $L_\beta = \mathrm{colim}_{\partial(\overrightarrow{\mathcal{R}}\downarrow\beta)} X$ is the empty limit hence being the initial object in $\mathcal{C}$. The map $X(\beta) \to Y(\beta)$ is just the relative latching map $X(\beta) \coprod_{L_\beta X} L_\beta Y \to Y(\beta)$ having the LLP wrt to every map in S.

Now we give a lifting problem

$$\begin{array}{ccc} L_\alpha X & \longrightarrow & E \\ \downarrow & & \downarrow \\ L_\alpha Y & \longrightarrow & B \end{array}$$

where $E \to B$ is a map in S. For β with $\deg(\beta) = 0$ appearing in $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$, we have a lifting $Y(\beta) \to E$ due to the analysis above. Since $\mathrm{colim}_{F^0\partial(\overrightarrow{\mathcal{R}}\downarrow\alpha)} X$ is just the coproduct of $X(\beta)$ with $\deg(\beta) = 0$, the lifting $\mathrm{colim}_{F^0\partial(\overrightarrow{\mathcal{R}}\downarrow\alpha)} Y \to E$ exists.

Now we prove this lemma by induction. Suppose $0 < k < \deg(\alpha)$ and there has existed a lifting $\mathrm{colim}_{F^{k-1}\partial(\overrightarrow{\mathcal{R}}\downarrow\alpha)} Y \to E$. Let $\beta \to \alpha$ be on object of $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ with $\deg(\beta) = k$ and then it induces a functor $\overrightarrow{\mathcal{R}} \downarrow \beta \to F^{k-1}\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ sending $\gamma \to \beta$ to $\gamma \to \beta \to \alpha$. This defines a map $L_\beta \to E$ which factors through $\mathrm{colim}_{F^{k-1}\partial(\overrightarrow{\mathcal{R}}\downarrow\alpha)} Y$. Then we have the following lifting problem

$$\begin{array}{ccc} X(\beta) \coprod_{L_\beta X} L_\beta Y & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y(\beta) & \longrightarrow & B \end{array}$$

which solves by assumption. This induces a map $Y(\beta) \to E$ with $\deg(\beta) = k$. Together with $Y(\beta) \to E$ for $\deg(\beta) < k$, we obtain $\mathrm{colim}_{F^k\partial(\overrightarrow{\mathcal{R}}\downarrow\alpha)} Y \to E$. Finally we will have $L_\alpha Y \to E$. □

Corollary A.4.22. *Let $\mathcal{R}$ be a Reedy category and $\mathcal{C}$ be a model category. Then a Reedy cofibration (resp. fibration) $f : X \to Y$ between $\mathcal{R}$-indexed diagrams, is an objectwise cofibration (resp. fibration).*

[30] [28, Lemma 15.3.9]

Proof. The map f is a Reedy cofibration if and only if the relative latching map $X(\alpha)\coprod_{L_\alpha X} L_\alpha Y \to Y(\alpha)$ is a cofibration i.e. having the LLP wrt all trivial fibrations. Then if f is a Reedy cofibration, from the lemma above $L_\alpha X \to L_\alpha Y$ will be a cofibration. $X(\alpha) \to Y(\alpha)$ is just the map $X(\alpha) \to X(\alpha)\coprod_{L_\alpha X} L_\alpha Y \to Y(\alpha)$ where the first map $X(\alpha) \to X(\alpha)\coprod_{L_\alpha X} L_\alpha Y$ is the pushout of $L_\alpha \to L_\alpha Y$ along $L_\alpha X \to X(\alpha)$ hence being a cofibration. Then $X(\alpha) \to Y(\alpha)$ will be the composition of two cofibrations hence being a cofibration as well. □

For Reedy categories $\mathcal{R}$, even though $\mathcal{C}^{\mathcal{R}}$ does not admit injective and projective model structures, derived functors of limits and colimits may exist.

Proposition A.4.23. [31] *Let $\mathcal{R}$ be a Reedy category and $\mathcal{C}$ be a model category.*

(1). If any constant $\mathcal{R}$-indexed diagram at a fibrant object of $\mathcal{C}$ is Reedy fibrant, then the colimit functor $\operatorname{colim} : \mathcal{C}^{\mathcal{R}} \to \mathcal{C}$ *is left Quillen.*

(2). If any constant $\mathcal{R}$-indexed diagram at a cofibrant object of $\mathcal{C}$ is Reedy cofibrant, then the limit functor $\lim : \mathcal{C}^{\mathcal{R}} \to \mathcal{C}$ *is right Quillen.*

There is a criterion to check whether a Reedy category $\mathcal{R}$ has the properties stated above.

Proposition A.4.24. [32] *Let $\mathcal{R}$ be a Reedy category.*

(1). It has fibrant constants if and only if the subcategory $\overleftarrow{\mathcal{R}}$ is a disjoint union of categories with a terminal object.

(2). It has cofibrant constants if and only if the subcategory $\overrightarrow{\mathcal{R}}$ is a disjoint union of categories with an initial object.

A.5 Enriched Model Categories

Definition A.5.1. A *simplicially enriched category $\mathcal{C}$* consists of the following data

(1). A set of objects $\mathrm{Ob}(\mathcal{C})$

(2). Any pair of objects (X, Y) in $\mathrm{Ob}(\mathcal{C})$ is associated with a simplicial set $\mathrm{Map}(X, Y)$. A 0-simplex in $\mathrm{Map}(X, Y)_0$ will be simply called a morphism from X to Y. The 1-simplex in $\mathrm{Map}(X, Y)_1$ will be called homotopies.

(3). For any triple of objects (X, Y, Z), there is a composition map of simplicial sets

$$\circ : \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z),\ (f, g) \mapsto g \circ f$$

(4) For any object X, there is an identity 0-simplex $\mathrm{id}_X \in \mathrm{Map}(X, X)_0$.

These data are required to satisfy usual associativity and unit axioms.

Remark A.5.2. We can associate every simplicially enriched category $\mathcal{C}$ two ordinary categories $\mathcal{C}_0$ and $\pi_0\mathcal{C}$, where

$$\mathrm{Hom}_{\mathcal{C}_0}(X, Y) = \mathrm{Map}(X, Y)_0, \text{ and } \mathrm{Hom}_{\pi_0\mathcal{C}}(X, Y) = \pi_0\mathrm{Map}(X, .Y)$$

For any morphism $f \in \mathrm{Map}(A, B)_0$ we can define a map of simplicial sets $f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ for any object X. Since in $\Delta\,[0]$ is the terminal object, for every simplicial set S there will exist a unique map $S_0 \to S_n$ for any n. For $x \in S_0$, we also denote its image in S_n by x. Then f^* sends $g \in \mathrm{Map}(B, X)_n$ to $g \circ f$ where $f \in \mathrm{Map}(A, B)_n$.

Therefore the composition in $\mathcal{C}_0$ is clear. And since the composition preserves homotopies, $\pi_0\mathcal{C}_0$ is clear as well.

[31] [56, Proposition 5.4.8]
[32] [17, Proposition 22.8]

Next we want to define the concept of *simplicial model categories* which is much more complicated than the simplicially enriched categories since we need to consider the compatibility of simplicial structure and model structure.

Definition A.5.3. Let $\mathcal{K},\ \mathcal{L},\ \mathcal{M}$ be categories. A triple of bifunctors

$$\otimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M},\ \ \mathrm{Map} : \mathcal{L}^{op} \times \mathcal{M} \to \mathcal{K},\ \ \{,\} : \mathcal{K}^{op} \times \mathcal{M} \to \mathcal{L}$$

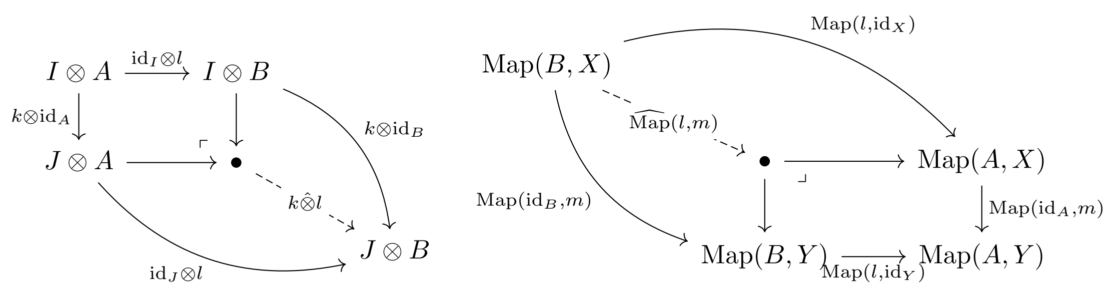
equipped with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(K \otimes L, M) \cong \mathrm{Hom}_{\mathcal{K}}(K, \mathrm{Map}(L, M)) \cong \mathrm{Hom}_{\mathcal{L}}(L, \{K, M\})$$

defines a *two-variable adjunction*. Note that we also write M^K for $\{K, M\}$.

Example A.5.4. A category $\mathcal{M}$ enriched over a *monoidal category* $\mathcal{V}$ is tensored and cotensored if the internal hom functor $\mathrm{Map} : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{V}$ is one of the adjoints of a two-variable adjunction in which $\mathcal{K} = \mathcal{V},\ \mathcal{L} = \mathcal{M}$. Then $\otimes$ is the tensor functor and $\{,\}$ is the cotensor functor.

Definition A.5.5 (The Leibniz Construction)**.** Given a bifunctor $\otimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ valued in a category with pushouts, the *Leibniz tensor* of a map $k : I \to J$ in $\mathcal{K}$ and a map $l : A \to B$ in $\mathcal{L}$ is the map $k\hat{\otimes}l$ induced by the pushout diagram.

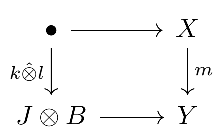


Given $l : A \to B$ in $\mathcal{L}$ and $m : X \to Y$ in $\mathcal{M}$, $\widehat{\mathrm{Map}}(l, m)$ is defined dually.

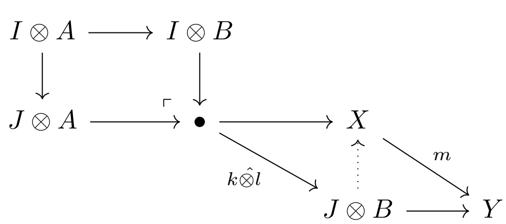
Proposition A.5.6. *Given classes $\mathcal{A},\ \mathcal{B},\ \mathcal{C}$ of maps in $\mathcal{K},\ \mathcal{L},\ \mathcal{M}$ respectively, then*

$$\mathcal{A}\hat{\otimes}\mathcal{B} \boxslash \mathcal{C} \Leftrightarrow \mathcal{B} \boxslash \widehat{\{\mathcal{A}, \mathcal{C}\}} \Leftrightarrow \mathcal{A} \boxslash \widehat{\mathrm{Map}}(\mathcal{B}, \mathcal{C})$$

Proof. We only prove $\mathcal{A}\hat{\otimes}\mathcal{B}\boxslash\mathcal{C} \Leftrightarrow \mathcal{A}\boxslash\widehat{\mathrm{Map}}(\mathcal{B}, \mathcal{C})$. Given maps $k : I \to I$ in $\mathcal{A}$, $l : A \to B$ in $\mathcal{B}$ and $m : X \to Y$ in $\mathcal{C}$, we want to solve the lifting problem



Looking at the following diagram



The map $\bullet \to X$ actually consists of a pair of maps $(J \otimes A \to X, I \otimes B \to X)$ which coincide on $I \otimes A$. Using the adjointness, it's equivalent to the commutative diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Map}(B,X) \\ \downarrow & & \downarrow \\ J & \longrightarrow & \mathrm{Map}(A,X) \end{array}$$

Again by the adjointness, the following two diagrams are equivalent

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow{\scriptstyle m} \\ J \otimes B & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} & & \mathrm{Map}(B,X) \\ & \nearrow & \downarrow \\ J & \longrightarrow & \mathrm{Map}(B,Y) \end{array}$$

Maps $J \to \mathrm{Map}(A,X)$ and $J \to \mathrm{Map}(B,Y)$ define a map $J \to \mathrm{Map}(B,Y) \coprod_{\mathrm{Map}(A,Y)} \mathrm{Map}(A,X)$. Hence the original lifting problem is equivalent to the following one

$$\begin{array}{ccccccc} I & \longrightarrow & \mathrm{Map}(B,X) & & & & \\ & \searrow & \uparrow & \searrow^{\widehat{\mathrm{Map}}(l,m)} & & & \\ & & J & \longrightarrow & \bullet & \longrightarrow & \mathrm{Map}(A,X) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathrm{Map}(B,Y) & \longrightarrow & \mathrm{Map}(A.Y) \end{array}$$

□

Definition A.5.7. A two-variable adjunction

$$\otimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}, \;\; \mathrm{Map} : \mathcal{L}^{op} \times \mathcal{M} \to \mathcal{K}, \;\; \{,\} : \mathcal{K}^{op} \times \mathcal{M} \to \mathcal{L}$$

between model categories $\mathcal{K}, \mathcal{L}, \mathcal{M}$ defines a *Quillen two-variable adjunction* if any, and hence all by proposition above, of the following statements is satisfied

(1). $\hat{\otimes}$ defines a map $Cof_{\mathcal{K}} \times Cof_{\mathcal{L}} \to Cof_{\mathcal{M}}$. Moreover if any one of the domain map is a weak equivalence, then the image will also be a weak equivalence.

(2). $\widehat{\mathrm{Map}}$ defines a map $Cof_{\mathcal{L}} \times Fib_{\mathcal{M}} \to Fib_{\mathcal{K}}$. The image is a weak equivalence if either of the domain map is.

(3). $\widehat{\{,\}}$ defines a map $Cof_{\mathcal{K}} \times Fib_{\mathcal{M}} \to Fib_{\mathcal{L}}$. The image is a weak equivalence if either of the domain map is.

Definition A.5.8. A *(closed symmetric) monoidal model category* is a (closed symmetric) monoidal category $(\mathcal{V}, \otimes, I)$ with a model structure so that the monoidal product $\otimes$ and the internal hom functor Map define a Quillen two-variable adjunction and furthermore maps $QI \otimes V \to I \otimes V \cong V$ and $V \otimes QI \to V \otimes I \cong V$ are weak equivalences if V is cofibrant, where QI is the cofibrant replacement of the identity object I with a weak equivalence $QI \to I$.

Remark A.5.9. If I is cofibrant, then the final requirement in the definition above is automatically satisfied and this comes from the following lemma.

Lemma A.5.10. *Given a Quillen two-variable adjunction as in the Definition A.5.7. Fix a cofibrant object A in $\mathcal{L}$, then the functor $- \otimes A : \mathcal{K} \to \mathcal{M}$ is a left Quillen functor.*

Proof. At first we see $- \otimes A$ is left adjoint to $\mathrm{Map}(A, -)$. Since $\iota_A : \emptyset \to A$ is a cofibration, for any map $k : I \to J$

$$\begin{array}{ccc} I \otimes \emptyset = \emptyset & \longrightarrow & I \otimes A \\ \downarrow & & \downarrow \\ J \otimes \emptyset = \emptyset & \longrightarrow & \bullet \end{array}$$

then the pushout will just be $I \otimes A$ itself. Note that $I \otimes \emptyset = \emptyset$ since the left adjoint functor $I \otimes -$ preserves colimits especially the empty colimit. Hence the map $k \hat{\otimes} \iota_A$ is just $k \otimes \mathrm{id}_A$. Then $- \otimes A$ preserves cofibrations and trivial fibrations by proposition above. □

Proof of Remark A.5.9. Since V is cofibrant, $- \otimes V$ is left Quillen. If I is cofibrant, $QI \to I$ will be a weak equivalence between cofibrant objects, via $- \otimes V$ it will be a weak equivalence by Lemma A.3.2. □

Definition A.5.11. If $\mathcal{V}$ is a monoidal model category, a *$\mathcal{V}$-model category* $\mathcal{M}$ is a $\mathcal{V}$-enriched model category which is tensored and cotensored in such a way that $(\otimes, \{,\}, \mathrm{Map})$ forms a Quillen two-variable adjunction and the map $QI \otimes M \to I \otimes M$ is a weak equivalence for every cofibrant object M in $\mathcal{M}$.

Example A.5.12. $s\mathbf{Set}$ is a closed symmetric monoidal model category. Its internal hom functor can be given in any category of presheaves and the tensor product is just the ordinary product. Then to have the adjointness it's necessary to have

$$\mathrm{Hom}_{s\mathbf{Set}}(\Delta^n \times X, Y) \cong \mathrm{Hom}_{s\mathbf{Set}}(\Delta^n, \mathrm{Map}(X, Y)) \cong \mathrm{Map}_n(X, Y)$$

Such internal hom functor is well defined and you can find details in [41, chapter I section 6]. It's a theorem in the theory of simplicial sets that they satisfy the condition of Quillen two-variable adjunction. See for example [14, Corollary 3.1.6 and Corollary 3.1.7].

Then a *simplicial model category* is just a $s\mathbf{Set}$-model category.

For a simplicial model category C we can define the *cosimplicial resolution* and *simplicial resolution* of an object X in $\mathcal{C}$ as follows. Let QX and PX be a cofibrant replacement and fibrant replacement of X respectively, then we define $\widetilde{X}^n = \Delta^n \otimes QX$ and $\widehat{X}_n = PX^{\Delta^n}$. [28, Proposition 16.1.3] shows this definition is equivalent to that in Definition A.4.19 using Reedy model categories.

For a $\mathcal{V}$-model category $\mathcal{M}$, passing to the level of homotopies, its homotopy theory $\mathrm{Ho}(\mathcal{M})$ has an enriched structure as well.

Theorem A.5.13. [33] *If $\mathcal{M}$ is a $\mathcal{V}$-model category, then* $\mathrm{Ho}(\mathcal{M})$ *is* $\mathrm{Ho}(\mathcal{V})$*-enriched with the total derived two-variable adjunction* $(\otimes^{\mathbb{L}}, \mathbb{R}\{,\}, \mathbb{R}\mathrm{Map})$.

Note that $V \otimes^{\mathbb{L}} M = QV \otimes QM$ where QV and QM are cofibrant replacements, and $\mathbb{R}\mathrm{Map}(X, Y) = \mathrm{Map}(QX, PY)$ where QX is a cofibrant replacement and PY is a fibrant replacement.

A.5.1 Weak Equivalences in a Simplicial Model Category

For a simplicial model category $\mathcal{C}$, we talk about some relationships between weak equivalences in $\mathcal{C}$ and weak equivalences in $s\mathbf{Set}$, which are useful for the next section about *left Bousfield localization*.

Proposition A.5.14. *Let $\mathcal{C}$ be a simplicial model category. If $g : X \to Y$ is a morphism in $\mathcal{C}$, then g is a weak equivalence if either of the following two conditions is satisfied.*

(1). For every fibrant object Z in $\mathcal{C}$, the map

$$g^* = \mathrm{Map}(g, \mathrm{id}_Z) : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a weak equivalence in $s\mathbf{Set}$.

[33] [31, Theorem 4.3.2].

(2). For every cofibrant object W, the map

$$g_* = \mathrm{Map}(\mathrm{id}_W, g) : \mathrm{Map}(W,X) \to \mathrm{Map}(W,Y)$$

is a weak equivalence in $s\mathbf{Set}$.

Proof. The two statements are dual and hence we only need to prove the first. First we replace X and Y by their fibrant replacements PX and PY with trivial cofibration $j_X : X \to PX$ and $j_Y : Y \to PY$. This will induce a map $Pg : PX \to PY$.

$$\begin{array}{ccc} \mathrm{Map}(PY,Z) & \xrightarrow{Pg^*} & \mathrm{Map}(PX,Z) \\ {\scriptstyle j_Y^*}\downarrow & & \downarrow{\scriptstyle j_X^*} \\ \mathrm{Map}(Y,Z) & \xrightarrow[g^*]{} & \mathrm{Map}(X,Z) \end{array}$$

j_Y* and j_X^* will then be trivial fibrations in $s\mathbf{Set}$ by the definition of simplicial model categories. g^* is a weak equivalence by assumption and then Pg^* is a weak equivalence as well. Especially there is an isomorphism

$$Pg_0^* : \pi_0\mathrm{Map}(PY,Z) \xrightarrow{\sim} \pi_0\mathrm{Map}(PX,Z)$$

Then there will exist a map $f : PY \to PX$ such that $f \circ Pg = \mathrm{id}_{PX}$ in $\pi_0\mathrm{Map}(PX,PX)$. $f_0^* = (Pg_0^*)^{-1}$ is an isomorphism for every fibrant object Z. Hence then there will exist some $h : PX \to PY$ such that $h \circ f = \mathrm{id}_{PY}$ in $\pi_0\mathrm{Map}(PY,PY)$. f is therefore an isomorphism in $\pi_0\mathcal{C}$ and so is Pg. According to the following lemma, Pg will be an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Next by Corollary A.2.23, Pg is weak equivalence. Finally g is hence a weak equivalence. □

Lemma A.5.15. *For a simplicial model category $\mathcal{C}$, the localization map $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ factors through $\pi_0\mathcal{C}$.*

Proof. We only need to prove any two maps $f, g \in \mathrm{Map}(X,Y)_0$ with a homotopy h from f to g define the same map in $\mathrm{Ho}(\mathcal{C})$. $h \in \mathrm{Map}(X,Y)_1$ is equivalent to a map $h : \Delta^1 \to \mathrm{Map}(X,Y)$ such that $hi_0 = f$, $hi_1 = g$ by Yoneda's lemma. By adjointness it's also equivalent to a map $\Delta^1 \otimes X \to Y$ which is denoted by h as well.

Next we replace X and Y by their fibrant replacements PX and PY with trivial cofibrations $j_X : X \to PX$ and $j_Y : Y \to PY$. Since every object in $s\mathbf{Set}$ is cofibrant, $\Delta^1 \otimes X \to \Delta^1 \otimes PX$ will then be a trivial cofibration.

$$\begin{array}{ccccc} X & \underset{i_1}{\overset{i_0}{\rightrightarrows}} & \Delta^1 \otimes X & \xrightarrow{h} & Y \\ {\scriptstyle j_X}\downarrow\wr & & \downarrow\wr & & \wr\downarrow{\scriptstyle j_Y} \\ PX & \underset{i_1}{\overset{i_0}{\rightrightarrows}} & \Delta^1 \otimes PX & \underset{Ph}{\dashrightarrow} & PY \end{array}$$

Since PY is fibrant, the arrow $Ph : \Delta^1 \otimes PX \to PY$ exists whose adjoint is $\widetilde{Ph} : PX \to PY^{\Delta^1}$. Let $f' = Ph \circ i_0$ and $g' = Ph \circ i_1$. Then $Ph \cdot i_0 \cdot j_X = j_Y \cdot h \cdot i_0$ which means $f' \cdot j_X = j_Y \cdot f$. Similarly $g' \cdot j_X = j_Y \cdot g$.

Since $(\partial_0^1, \partial_1^1) : \Delta^0 \coprod \Delta^0 \to \Delta^1$ is a cofibration and PY is fibrant,

$$ev = (ev_0, ev_1) : PY^{\Delta^1} \to PY^{\Delta^0 \coprod \Delta^0} = PY \times PY$$

is a fibration

$$\begin{array}{ccccc} & \overset{\widetilde{Ph}}{\nearrow} & PY^{\Delta^1} & \overset{s}{\nwarrow} & \\ & & \downarrow{\scriptstyle ev} & & \\ PX & \xrightarrow[(f',g')]{} & PY \times PY & \longleftarrow & PY \\ & & \downdownarrows & & \\ & & PY & & \end{array}$$

In the diagram above s is induce by the map $\Delta^1 \to \Delta^0$. Since $\partial_i^1 : \Delta^0 \to \Delta^1$ is a trivial cofibration, ev_i will then be a trivial fibration. $ev_i \circ s = \mathrm{id}_{PY}$. Hence s is a weak equivalence. All of these prove PY^{Δ^1} is a path object and $\widetilde{Ph}$ is a right homotopy from f' to g'. Then in $\mathrm{Ho}(\mathcal{C})$, $[f'] = [g']$. But then

$$[f] = [j_Y]^{-1}[f'][j_X] = [j_Y]^{-1}[g'][j_X] = [g]$$

□

Proposition A.5.16. *If moreover we suppose $g : X \to Y$ is a morphism between cofibrant objects in a simplicial model category $\mathcal{C}$, then g is a weak equivalence if and obly if for all fibrant obejct Z, the map $g^* : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a weak equivalence in $s\mathbf{Set}$. The dual statement is true as well.*

Proof. The part "$\Leftarrow$" has been proved above.

"$\Rightarrow$": Since g is a weak equivalence, g will be an isomorphism in $\mathrm{Ho}(\mathcal{C})$. According to Theorem A.5.13, $g^* : \mathbb{R}\mathrm{Map}(Y, Z) \to \mathbb{R}\mathrm{Map}(X, Z)$ is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$. But for any cofibrant object X and fibration object Z, $\mathbb{R}\mathrm{Map}(X, Z) = \mathrm{Map}(X, Z)$. Then g^* is an isomorphism between $\mathrm{Map}(Y, Z)$ and $\mathrm{Map}(X, Z)$ in $\mathrm{Ho}(s\mathbf{Set})$. Therefore g^* is a weak euivalence in $s\mathbf{Set}$ by Corollary A.2.23. □

A.5.2 Local Homotopy Limits and Colimits

Finally we talk about the local version of homotopy limits and colimits in a simplicial model category $\mathcal{C}$, which is due to Bousfield and Kan.

Definition A.5.17. Let I be a small category and $\mathcal{C}$ be a simplicial model category.

(1). If X is an I-indexed diagram i.e. $X : I \to \mathcal{C}$ and $K : I^{op} \to s\mathbf{Set}$, then the coequalizer of the following diagram

$$\coprod_{\sigma:\alpha\to\beta\in I} K(\beta) \otimes X(\alpha) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{\alpha\in I} K(\alpha) \otimes X(\alpha)$$

is denoted by $K \otimes_I X$ and called the *functor tensor product* where the component of ϕ on $\sigma : \alpha \to \beta$ is $\mathrm{id}_{K(\beta)} \otimes \sigma_* : K(\beta) \otimes X(\alpha) \to K(\beta) \otimes X(\beta)$ and the component of ψ on σ is $\sigma^* \otimes \mathrm{id}_{X(\alpha)} : K(\beta) \otimes X(\alpha) \to K(\alpha) \otimes X(\beta)$.

(2). If $I = \Delta^{op}$ and $K = \triangle : \Delta \to s\mathbf{Set}$ is the Yoneda embedding, then the *geometric realization* $|X|$ of X is defined to be $\triangle \otimes_{\Delta^{op}} X$, i.e. the coequalizer of the following diagram

$$\coprod_{\sigma:[k]\to[n]\in\Delta} \Delta^k \otimes X_n \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{[n]\in\Delta} \Delta^n \otimes X_n$$

where the component of ϕ on $\sigma : [k] \to [n]$ is $\mathrm{id}_{\Delta^k} \otimes \sigma^* : \Delta^k \otimes X_n \to \Delta^k \otimes X_k$ and the component of ψ on σ is $\sigma_* \otimes \mathrm{id}_{X_n} : \Delta^k \otimes X_n \to \Delta^n \otimes X_n$.

(3). If X is a I-indexed diagram and $K : I \to s\mathbf{Set}$, then the equalizer of the following diagram

$$\prod_{\alpha\in I} X(\alpha)^{K(\alpha)} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\sigma:\alpha\to\beta} X(\beta)^{K(\alpha)}$$

is denoted by $\{K, X\}^I$ and called the *functor cotensor product* where the projection of ϕ on the component $\sigma : \alpha \to \beta$ is $\sigma_*^{\mathrm{id}_{K(\alpha)}} : X(\alpha)^{K(\alpha)} \to X(\beta)^{K(\alpha)}$ and the projection of ψ on σ is $(\mathrm{id}_{X(\beta)})^{\sigma_*} : X(\beta)^{K(\beta)} \to X(\beta)^{K(\alpha)}$.

(4). If $I = \Delta$ and $K = \triangle$ is the Yoneda embedding , then the *total object* $\mathrm{Tot}X$ of $X : \Delta \to \mathcal{C}$ is difined to be $\{\triangle, X\}^{\Delta}$ i.e. the equalizer of the following diagram

$$\prod_{[n]\in\Delta}(X^n)^{\Delta^n} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\sigma:[n]\to[k]}(X^k)^{\Delta^n}$$

where the projection of ϕ on the component $\sigma : [n] \to [k]$ is $\sigma_*^{\mathrm{id}_{\Delta^n}} : (X^n)^{\Delta^n} \to (X^k)^{\Delta^n}$ and the projection of ψ on σ is $(\mathrm{id}_{X^k})^{\sigma_*} : (X^k)^{\Delta^k} \to (X^k)^{\Delta^n}$.

Remark A.5.18. $K \otimes_I X$ is actually the *coend*[34] $\int^{\alpha\in I} K(\alpha)\otimes X(\alpha)$ and dually $\{K, X\}^I$ is the *end* $\int_{\alpha\in I}\{K(\alpha), X(\alpha)\} = \int_{\alpha\in I} X(\alpha)^{K(\alpha)}$.

Example A.5.19. A classical result says if $K = * : I^{op} \to s\mathbf{Set}$ sends every object of I to the terminal object in $s\mathbf{Set}$, then $* \otimes_I X \cong \mathrm{colim}_I X$. It's just the proof that a category with coproducts and coeuqlizers is cocomplete. Therefore we can view $K \otimes_I X$ as the *weighted colimit* where the *weight* is the functor $K : I^{op} \to s\mathbf{Set}$. Dually $\{K, X\}^I$ will be the *weighted limit*. In the following we will show homotopy limits and colimits are special cases of this type.

If we replace I by any Reedy category $\mathcal{R}$, we will obtain the following theorem.

Theorem A.5.20. [35] *Let $\mathcal{R}$ be a Reedy category and $\mathcal{C}$ be a simplicial model category. Then the functor tensor product*

$$\otimes_{\mathcal{R}} : s\mathbf{Set}^{\mathcal{R}^{op}} \times \mathcal{C}^{\mathcal{R}} \to \mathcal{C},\ \ (K, X) \mapsto K \otimes_{\mathcal{R}} X := \int^{\alpha\in\mathcal{R}} K(\alpha) \otimes X(\alpha)$$

is a left Quillen bifunctor with respect to the Reedy model structures, and dually the functor cotensor product

$$\{,\}^{\mathcal{R}} : (s\mathbf{Set}^{\mathcal{R}})^{op} \times \mathcal{C}^{\mathcal{R}} \to \mathcal{C},\ \ (K, X) \mapsto \{K, X\}^{\mathcal{R}} := \int_{\alpha\in\mathcal{R}} X(\alpha)^{K(\alpha)}$$

is a right Quillen bifunctor with respect to the Reedy model structures.

Remark A.5.21. If I is a small category and $\mathcal{C}^I$ can be equipped with injective or projective model structure, then the theorem above is true as well, which means it only depends on objectwise cofirations or objectwise fibrations. See Theorem 18.4.1 in [28].

For any small category I, we can define a functor $I^{op} \to s\mathbf{Set}$ such that it sends an object α of I^{op} to the simplicial set $N(\alpha \downarrow I)$ where $N : \mathbf{Cat} \to s\mathbf{Set}$ is the nerve functor.

Definition A.5.22. [36] Let I be a small category and $\mathcal{C}$ be a simplicial model category. Suppose $X : I \to \mathcal{C}$ is an I-indexed diagram on $\mathcal{C}$. Then

(1). the *homotopy colimit* of X is defined to be

$$\mathrm{hocolim}X := N(- \downarrow I) \otimes_I QX$$

(2). the *homotopy limit* of X is defined to be

$$\mathrm{holim}X := \{N(I \downarrow -), PX\}^I$$

where QX and PX are weakly equivalent replacements of X such that their values are in cofibrant objects and fibrant objects respectively. If $\mathcal{C}$ is functorial, then Q and P exist in an obvious way.

[34] See [40, Section IX.6]

[35] [55, Theorem 14.3.1] or [28, Theorem 18.4.11]

[36] [59, Definition 8.2]

Remark A.5.23. In [28], $N(-\downarrow I)^{op}$ is chosen to define the homotopy colimit. Up to homotopy, two choices will derive the same result, since the nerve of a category is weakly equivalent to that of its opposite. More discussions can be found in [28, Remark 18.1.11]. Actually $N(-\downarrow I)^{op}$ and $N(-\downarrow I)$ are all cofibrant replacements of the terminal object in $s\mathbf{Set}^{I^{op}}_{proj}$ [28, Proposition 14.8.9] and choosing any other cofibrant replacement as the weight functor will obtain weakly equivalent result [28, Theorem 19.4.6].

Remark A.5.24. This definition for homotopy colimits and limits are equivalent to previous global one using derived functors. In [55], Emily Riehl introduces the concept of *two-sided simplicial bar resolution* as follows. For $X : I \to \mathcal{C}$ and $K : I^{op} \to s\mathbf{Set}$ we can define a simplicial object $B_\bullet(K, I, X)$ in $\mathcal{C}$ such that

$$B_n(K, I, X) = \coprod_{d:[n]\to I} K(d_n) \otimes X(d_0)$$

and the *bar construction* is defined to be the geometric realization of this simplicial bar resolution i.e.

$$B(K, I, X) := |B_\bullet(K, I, X)| = \triangle \otimes_{\Delta^{op}} B_\bullet(K, I, X)$$

For any object α of I, there is a functor $I^{op} \to s\mathbf{Set}$ such that it sends any object β of I to the constant simplicial set $\mathrm{Hom}_I(\beta, \alpha)$. We denote this functor just by α. Then this defines a functor

$$B(I, I, X) : I \to \mathcal{C}, \ \ \alpha \mapsto B(\alpha, I, X)$$

If $\mathcal{C}$ has a functorial cofibrant replacement functor Q, then $B(I, I, Q-) : \mathcal{C}^I \to \mathcal{C}^I$ will be the left deformation (see Remark A.3.4) for $\mathrm{colim} : \mathcal{C}^I \to \mathcal{C}$. Then the left derived functor of colim exists which is just $B(*, I, Q-)$ [55, Corollary 5.1.3]. In [55, Theorem 6.6.1] the author proves $B(*, I, X) \cong N(-\downarrow I) \otimes_I X$. Therefore the definition of homotopy colimits above is equivalent to that using the left derived functor.

A dual discussion for homotopy limits is valid as well. The reader can also find the same discussion in [59, Section 8].

There are some beautiful corollaries for constructions we discussed above one of which is that homotopy limits and colimits preserve weak equivalences in a certain class of objects, which are similar to ordinary limits and colimits preserving isomorphisms.

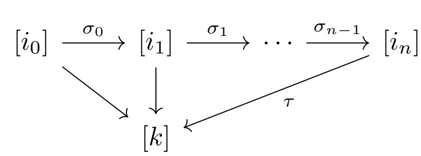
Corollary A.5.25. [37] *Let I be a small category and $\mathcal{C}$ be a simplicial model category.*

(1). If $f : X \to Y$ is a map of I-indexed diagrams in $\mathcal{C}$ such that for any object α of I, f_α is a weak equivalence between cofibrant objects, then the induced map $f_* : \mathrm{hocolim}X \to \mathrm{hocolim}Y$ is a weak equivalence of cofibrant objects.

(2). If $f : X \to Y$ is a map of I-indexed diagrams in $\mathcal{C}$ such that for any object α of I, f_α is a weak equivalence between fibrant objects, then the induced map $f^* : \mathrm{holim}X \to \mathrm{holim}Y$ is a weak equivalence of fibrant objects.

Proof. It just follows from the definition of Quillen bifunctors and Ken Brown's lemma. □

Definition A.5.26. The *Bousfield-Kan map of cosimplicial simplicial sets* is the map $\phi : N(\Delta \downarrow -) \to \triangle$ such that ϕ_k sends the following n-simplex



to the n-simplex

$$[\tau\sigma_{n-1}\cdots\sigma_0(i_0), \tau\sigma_{n-1}\cdots\sigma_1(i_1), \cdots, \tau\sigma_{n-1}(i_{n-1}), \tau(i_n)]$$

of Δ^k which is just a map $[n] \to [k]$.

[37] [28, Theorem 18.5.3]

Proposition A.5.27. *The Bousfield-Kan map of cosimplicial simplicial set is a weak equivalence of Reedy cofibrant cosimplicial simplicial sets.*

Proof. It's [28, Proposition 18.7.2]. □

Definition A.5.28. Let $\mathcal{C}$ be a simplicial model category.

(1). If X is a simplicial object in $\mathcal{C}$, then the *Bousfield-Kan map* is the map

$$N(- \downarrow \Delta^{op})^{op} \otimes_{\Delta^{op}} X \xrightarrow{\phi \otimes_{\Delta^{op}} \mathrm{id}_X} \triangle \otimes_{\Delta^{op}} X = |X|$$

(2). If X is a cosimplicial object in $\mathcal{C}$, then *Bousfield-Kan map* is the map

$$\mathrm{Tot}X = \{\triangle, X\}^{\Delta} \xrightarrow{(\mathrm{id}_X)^{\phi}} \{N(\Delta \downarrow -), X\}^{\Delta}$$

Theorem A.5.29. *Let $\mathcal{C}$ be a simplicial model category.*

(1). If X is a Reedy cofibrant simplicial object in $\mathcal{C}$, then the Bousfield-Kan map $\mathrm{hocolim}X \to |X|$ *is a natural weak equivalence.*

(2). If X is a Reedy fibrant cosimplicial object in $\mathcal{C}$, then the Bousfield-Kan map $\mathrm{Tot}X \to \mathrm{holim}X$ *is a natural weak equivalence.*

Proof. Since Reedy cofibrations (resp. fibrations) are objectwise cofibrations (resp. fibrations), the left side of morphisms in the definition above are homotopy colimits (resp. limits). Then this theorem follows from the proposition above and Quillen bifunctors for Reedy model categories i.e. Theorem A.5.20. □

Corollary A.5.30. *If X is a simplicial simplicial set i.e. $\Delta^{op} \to s\mathbf{Set}$, then the Bousfield-Kan map* $\mathrm{hocolim}X \to |X|$ *is a weak equivalence.*

Proof. [28, Corollary 15.8.8] says any simplicial object in $s\mathbf{Set}$ is Reedy cofibrant. □

A.6 Bousfield Localization

We first deal with the case when $\mathcal{C}$ is an ordinary category and then pass to simplicial model categories.

A.6.1 Localization for Ordinary Categories

Suppose $\mathcal{C}$ is an ordinary category and $S \subseteq \mathcal{C}$ is a subcategory of $\mathcal{C}$ containing all isomorphisms. Morphisms in S are called *equivalences*.

Definition A.6.1. An object X of $\mathcal{C}$ is said to be *S-local* if for any equivalence $f : A \to B$ in S, the induce map

$$f^* : \mathrm{Hom}_{\mathcal{C}}(B, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, X)$$

is a bijection.

Roughly speaking X is S-local if objects in S can not be distinguished by mapping them into X.

Definition A.6.2. A map $\mu_Y : Y \to X$ in $\mathcal{C}$ is called an *S-localization* of Y if X is local and μ_Y is an equivalence i.e. μ_Y belongs to S. The pair $(\mathcal{C}, S)$ is said to have *good localizations* if every object in $\mathcal{C}$ has a localization.

Remark A.6.3. Supose $\mu_Y : Y \to X$ and $\mu_{Y'} : Y' \to X'$ are localizations of Y and Y'. Then for any map $f : Y \to Y'$, there will exist a unique map $g : X \to X'$ such that $g \cdot \mu_Y = \mu_{Y'} \cdot f$.

$$\begin{CD} Y @>{\mu_Y}>> X \\ @V{f}VV @VV{\exists!\ g}V \\ Y' @>>{\mu_{Y'}}> X' \end{CD}$$

Proof. Since $\mu_Y \in S$ and X' is local,

$$\mu_Y^* : \mathrm{Hom}_{\mathcal{C}}(X, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X')$$

Then there will exist a unique $g \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ satisfying $\mu_Y^*(g) = g \cdot \mu_Y = \mu_{Y'} \cdot f$. □

Corollary A.6.4. *Any two localizations of an object Y in $\mathcal{C}$ are canonically isomorphic.*

Remark A.6.5. If $(\mathcal{C}, S)$ has good localizations, then there will exist a lozalization functor L_S sending an object Y in S to a localization $\mu_Y : Y \to X$ and any morphism $f : Y \to Y'$ to the commutative diagram

$$\begin{CD} Y @>{\mu_Y}>> X \\ @V{f}VV @VV{g}V \\ Y' @>>{\mu_{Y'}}> X' \end{CD}$$

Any different choices of localizations will obtain isomorphic localization functors. For simplicity we just write $L_S(Y) = X$ and $L_S(f) = g$. In this way we define a functor $L_S : \mathcal{C} \to Loc_S(\mathcal{C})$ where the latter is the full category of $\mathcal{C}$ consisting of local objects.

Proposition A.6.6. *If $f : Y \to Y'$ is an equivalence i.e. in S, then $L_S(f)$ is an isomorphism. Moreover if S satisfies the two-of-three property, then the converse is true as well.*

Proof. If S satisfies the two-of-three property, the converse will be clear since the isomorphism $L_S(f) \in S$ and then $\mu_{Y'} \cdot f = L_S(f) \cdot \mu_Y \in S$ implies $f \in S$.

Now suppose $f \in S$. Then $\mu_{Y'} \cdot f \in S$. Since X is local, we have a bijection

$$(\mu_{Y'} \cdot f)^* : \mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

and then there will exist a unique map $g' : X' \to X$ satisfying $g' \cdot \mu_{Y'} \cdot f = \mu_Y$. Hence $gg'\mu_{Y'}f = g\mu_Y = \mu_{Y'}f$

$$\begin{CD} Y @>{\mu_{Y'} \cdot f}>> X' \\ @V{\mu_Y}VV @VV{\exists!\ g'}V \\ X @= X \\ @V{g}VV @VV{g}V \\ X' @= X' \end{CD} \qquad (\mathrm{id}_{X'} : X' \to X')$$

Then $L_S(g \cdot \mu_Y) = g \cdot g' = \mathrm{id}_{X'}$ when choosing $\mu_Y \cdot f$ as the localization of Y. On the other hand

$$\begin{CD} Y @>{\mu_Y}>> X \\ @V{\mu_{Y'} \cdot f}VV @VV{g}V \\ X' @= X' \\ @V{g'}VV @VV{g'}V \\ X @= X \end{CD} \qquad (\mu_Y : Y \to X,\ \mathrm{id}_X : X \to X)$$

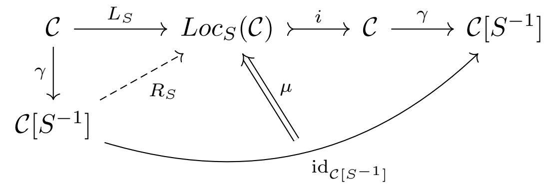
when choosing μ_Y as the localization of Y, we have $L_S(\mu_Y) = g' \cdot g = \mathrm{id}_X$. This means g is an isomorphism. □

Theorem A.6.7. *Suppose $(\mathcal{C}, S)$ has good localizations. Then*

$$Loc_S(\mathcal{C}) \xrightarrow{i} \mathcal{C} \xrightarrow{\gamma} \mathcal{C}[S^{-1}]$$

is an equivalence between $Loc_S(\mathcal{C})$ and $\mathcal{C}[S^{-1}]$, where i is the embedding functor and γ is the localization functor.

Proof. Since $L_S : \mathcal{C} \to Loc_S(\mathcal{C})$ sned all morphisms in S to isomorphisms, there exists a functor $R_S : \mathcal{C}[S^{-1}] \to Loc_S(\mathcal{C})$ such that $R_S \circ \gamma = L_S$.



$\gamma \cdot i \cdot L_S$ sends any object Y in $\mathcal{C}$ to its localization X. Then $(\mu_Y)_{Y \in \mathrm{Ob}(\mathcal{C})}$ gives a natural isomorphism between $\mathrm{id}_{\mathcal{C}}[S^{-1}]$ and $\gamma \cdot i \cdot R_S$. On the other hand, for simplicity we may choose the localization of a local object X to be itself and then $R_S \cdot \gamma \cdot i = L_S \cdot i = \mathrm{id}_{Loc_S(\mathcal{C})}$. □

A.6.2 Localization for Simplicial Model Categories

Now we deal with the localization for model categories.

Definition A.6.8. Suppose $\mathcal{C}$ is a model category. Then a *left Bousfield localization* $\mathcal{C}_{loc}$ of $\mathcal{C}$ is another model category with the same underlying categorical structure such that $Cof_{\mathcal{C}_{loc}} = Cof_{\mathcal{C}}$ but $W_{\mathcal{C}} \subseteq W_{\mathcal{C}_{loc}}$.

Remark A.6.9. By definition in $\mathcal{C}_{loc}$ there are less fibrations $Fib_{\mathcal{C}_{loc}} \subseteq Fib_{\mathcal{C}}$. And we obtain a Quillen pair $\mathrm{id} : \mathcal{C} \rightleftarrows \mathcal{C}_{loc} : \mathrm{id}$, where the left identity functor preserves cofibrations and weak equivalences.

In the following we assume $\mathcal{C}$ is a simplicial model category and S is a class of morphisms in it. Then there is a right derived internal hom functor

$$\mathbb{R}\mathrm{Map} : \mathrm{Ho}(\mathcal{C})^{op} \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(s\mathbf{Set}), \ \ (X, Y) \mapsto \mathrm{Map}(QX, PY)$$

where QX is a cofibratn replacement and PY is a fibrant replacement.

Definition A.6.10. (1). An object X of $\mathcal{C}$ is *S-locl* is for all $f : A \to B$ in S, the induced map $f^* : \mathbb{R}\mathrm{Map}(B, X) \to \mathbb{R}\mathrm{Map}(A, X)$ is an isomorphism in $\mathrm{Ho}(s\mathbf{Set})$, which is equivalent to saying $Qf^* : \mathrm{Map}(QB, PX) \to \mathrm{Map}(QA, PX)$ is a weak equivalence in $s\mathbf{Set}$.[38]

(2). A map $g : X \to Y$ in $\mathcal{C}$ is an *S-equivalence* if for every S-local object Z, the induced morphism $Qg^* : \mathrm{Map}(QY, PZ) \to \mathrm{Map}(QX, PZ)$ is a weak equivalence in $s\mathbf{Set}$. The class of S-equivalences is denoted by W_S.

(3). An *S-localization* of an object X in $\mathcal{C}$ is an S-equivalence $X \to \hat{X}$ where $\hat{X}$ is S-local.

Lemma A.6.11. *Every weak equivalence in $\mathcal{C}$ is an S-equivalence which means $W \subseteq W_S$.*

Proof. A map $g : X \to Y$ in $\mathcal{C}$ is a weak equivalence if and only if $Qg : QX \to QY$ is a weak equivalence. By Proposition A.5.16, for a weak equivalence Qg and a fibrant object Z, the induce morphism $Qg^* : \mathrm{Map}(QY, Z) \to \mathrm{Map}(QX, Z)$ is a weak equivalence in $s\mathbf{Set}$. This proves g is an S-equivalence. □

Lemma A.6.12. *If $g : X \to X'$ is a weak equivalence, then X is S-local if and only if X' is S-local.*

[38] From Proposition A.5.16, this definition is independent from the choice of cofibrant replacements and fibrant replacements.

Proof. For a map $f : A \to B$ in S, we have

$$\begin{array}{ccc} \mathrm{Map}(QB, PX) & \xrightarrow{\sim} & \mathrm{Map}(QB, PX') \\ \downarrow & & \downarrow \\ \mathrm{Map}(QA, PX) & \xrightarrow{\sim} & \mathrm{Map}(QA, PX') \end{array}$$

□

Lemma A.6.13. *An S-equivalence $f : X \to Y$ between S-local objects is a weak equivalence in $\mathcal{C}$.*

Proof. Since X is S-local, the map $Qf^* : \mathrm{Map}(QY, PX) \to \mathrm{Map}(QX, PX)$ is a weak equivalence between simplicial sets. Then especially

$$\pi_0 Qf^* : \pi_0\mathrm{Map}(QY, PX) \xrightarrow{\sim} \pi_0\mathrm{Map}(QX, PX)$$

is an isomorphism at the level of path components. Here we have a sequence

$$\emptyset \rightarrowtail QX \overset{p_X}{\underset{\sim}{\twoheadrightarrow}} X \overset{j_X}{\underset{\sim}{\rightarrowtail}} PX \twoheadrightarrow *$$

There exists a map $g : QY \to PX$ such that $g \cdot Qf = j_X \cdot p_X$ in $\pi_0\mathrm{Map}(QX, PX)$. By Lemma A.5.15, $[g \cdot Qf] = [j_X \cdot p_X]$ in $\mathrm{Ho}(\mathcal{C})$.

Next since Y is S-local,

$$Qf^* : \mathrm{Map}(QY, PY) \to \mathrm{Map}(QX, PY)$$

is a weak equivalence. And then we have $Qf^*(j_Y \cdot p_Y) = j_Y \cdot p_Y \cdot Qf = Pf \cdot j_X \cdot p_X = Pf \cdot g \cdot Qf$ in $\pi_0\mathrm{Map}(QX, PY)$. But since $\pi_0 Qf^*$ is an isomorphism, $j_Y \cdot p_Y = Pf \cdot g$ in $\pi_0\mathrm{Map}(QY, PY)$. Then $[j_Y \cdot p_Y] = [Pf \cdot g]$ in $\mathrm{Ho}(\mathcal{C})$. This means $[g]$ has a left inverse and a right inverse, hence an isomorphism, in $\mathrm{Ho}(\mathcal{C})$ and therefore $[Qf]$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Qf is then a weak equivalence and so is f. □

Theorem A.6.14. *If $\mathcal{C}$ is left proper combinatorial simplicial model category and S is a small set of morphisms in $\mathcal{C}$, then the left Bousfield localization $L_S\mathcal{C}$ with respect to W_S of S-equivalences exists and is itself a left proper combinatorial simplicial model category. Moreover fibrant objects in $L_S\mathcal{C}$ are precisely the S-local objects of $\mathcal{C}$ that are fibrant in $\mathcal{C}$.*

Proof. See [6, Theorem 4.7] or [1, Theorem 4.1]. □

Remark A.6.15. Conversely every left Bousfield localization can be obtained in this way. Details can be found in [1, Proposition 3.10].

Remark A.6.16. There is a Quillen adjunction $\mathrm{id} : \mathcal{C} \rightleftarrows L_S\mathcal{C} : \mathrm{id}$ and passing to homotopy theories they form an adjoint pair $\mathbb{L}\mathrm{id} : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(L_S\mathcal{C}) : \mathbb{R}\mathrm{id}$. $\mathbb{R}\mathrm{id} = \mathrm{id} \circ P$ where P is the fibrant replacement functor of $L_S\mathcal{C}$. Since fibrant (resp. cofibrant) objects in $L_S\mathcal{C}$ are fibrant (resp. cofibrant) in $\mathcal{C}$, $\mathbb{R}\mathrm{id}$ is then fully faithful. At first we see its image in $\mathcal{C}$ consists of fibrant objects which are S-local as well. For any S-local object in $\mathcal{C}$, it's weakly equivalent to its fibrant replacement. From Lemma A.6.12, its fibrant replacement is S-local as well. Therefore S-local objects are in the essential image of $\mathbb{R}\mathrm{id}$. If X is isomorphic to an S-local object in $\mathrm{Ho}(\mathcal{C})$ whose internal hom functor is $\mathbb{R}\mathrm{Map}$, by the definition of S-local objects, it's then clear to see X will be S-local. Therefore the essential image of the fully faithful functor $\mathbb{R}\mathrm{id}$ consists of S-local objects.

We know in the Bousfield localization $L_S\mathcal{C}$, trivial firbations are just those trivial fibrations in the original category $\mathcal{C}$ but fibrations are different. However, the following theorem will tell us fibrations will also not be changed in the local world.

Theorem A.6.17. *Suppose X and Y are fibrant objects in $L_S\mathcal{C}$. Then a map $p : X \to Y$ in $L_S\mathcal{C}$ is a fibration if and only if it's a fibration in $\mathcal{C}$.*

Proof. We only need to prove a fibration $p : X \to Y$ in $\mathcal{C}$ is a fibration in $L_S\mathcal{C}$ when X and Y are fibrant in $L_S\mathcal{C}$. Now given an arbitrary lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where $A \to B$ is a trivial cofibration i.e. a cofibration which is also an S-equivalence. Next we decompose this lifting problem in $L_S\mathcal{C}$ as the following one.

$$\begin{array}{ccccc} A & \overset{\sim}{\rightarrowtail} & A' & \twoheadrightarrow & X \\ {\scriptstyle\sim}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ B & \underset{\sim}{\rightarrowtail} & B' & \twoheadrightarrow & Y \end{array}$$

Then A' and B' will be fibrant in $L_S\mathcal{C}$ especially S-local. We also decompose the morphism $A' \to B'$ as a composition of a fibration and a trivial cofibration in $L_S\mathcal{C}$.

$$\begin{array}{ccccc} A & \overset{\sim}{\rightarrowtail} & A' & \twoheadrightarrow & X \\ \| & & {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle p} \\ A & \overset{\sim}{\rightarrowtail} & C & \twoheadrightarrow & Y \\ {\scriptstyle\sim}\downarrow & & \downarrow & & \| \\ B & \overset{\sim}{\rightarrowtail} & B' & \twoheadrightarrow & Y \end{array}$$

Since A' and C are S-local, by Lemma A.6.13 the S-equivalence $A' \to C$ will be a weak equivalence in $\mathcal{C}$. Then $A' \to C$ is a trivial cofibration in $\mathcal{C}$. The lifting $C \to X$ exists. And moreover the lifting $B \to C$ exists in the model category $L_S\mathcal{C}$. The composition gives a solution to the original lifting problem. □

References

[1] nLab page, Left Bousfield localization of model categories.

[2] nLab page, Model structure on dg-algebras.

[3] *Stacks Project*. `https://stacks.math.columbia.edu/`.

[4] Jiří Adámek and Jiří Rosickỳ. *Locally presentable and accessible categories*, volume 189. Cambridge University Press, 1994.

[5] Jarod Alper. *Notes on Stacks and Moduli*. Lecture page, March 11, 2022.

[6] Clark Barwick. On left and right model categories and left and right Bousfield localizations. *Homology, Homotopy and Applications*, 1(1):1–76, 2010.

[7] David Ben-Zvi and David Nadler. Loop spaces and connections, arXiv:1002.3636. *Journal of Topology*, 5(2):377–430, 2012.

[8] Siegfried Bosch. *Algebraic Geometry and Commutative Algebra*. Springer, 2013.

[9] Aldridge Knight Bousfield and Victor KAM Gugenheim. *On PL DeRham Theory and Rational Homotopy Type*, volume 179. American Mathematical Soc., 1976.

[10] Lawrence Breen. *On the Classification of* 2*-Gerbes and* 2*-Stacks*. Société Mathématique de France, 1994.

[11] Damien Calaque. Three lectures on derived symplectic geometry and topological field theories. *Indagationes Mathematicae*, 25(5):926–947, 2014.

[12] Alberto S. Cattaneo, Domenico Fiorenza, and Riccardo Longoni. Graded poisson algebras, arXiv:1811.07395v2. 2018.

[13] Ionuţ Ciocan-Fontanine and Mikhail Kapranov. Derived Quot schemes, arXiv:math/9905174 . In *Annales Scientifiques de l'École Normale Supérieure*, volume 34, pages 403–440. Elsevier, 2001.

[14] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*, volume 180. Cambridge University Press, 2019.

[15] Martin Doubek, Martin Markl, and Petr Zima. Deformation theory (lecture notes), arXiv:0705.3719. 2007.

[16] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 136, pages 9–51. Cambridge University Press, 2004.

[17] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy Limit Functors on Model Categories and Homotopical Categories*. Number 113. American Mathematical Soc., 2004.

[18] William G. Dwyer and Daniel M. Kan. Calculating simplicial localizations. *Journal of Pure and Applied Algebra*, 18(1):17–35, 1980.

[19] David Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*, volume 150. Springer Science & Business Media, 1995.

[20] David Eisenbud and Joe Harris. *3264 AND ALL THAT: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[21] Peter Gabriel and Michel Zisman. *Calculus of Fractions and Homotopy Theory*. Springer-Verlag New York Inc., 1967.

[22] Jean Giraud. *Cohomogie non abélienne*. Springer-Verlag Berlin Heidelberg New York, 1971.

[23] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Springer Science & Business Media, 2009.

[24] Paul G. Goerss and Kristen Schemmerhorn. Model categories and simplicial methods, arXiv:math/0609537. 2006.

[25] Alexander Grothendieck. Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. *Publications Mathématiques de l'IHÉS*, 28:5–255, 1966.

[26] Alexander Grothendieck. *À la Poursuite des Champs*. English version: arXiv:2111.01000, 1983.

[27] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[28] Philip S. Hirschhorn. *Model Categories and Their Localizations*. Number 99. American Mathematical Soc., 2003.

[29] Sharon Hollander. A homotopy theory for stacks, arXiv:math/0110247. *Israel Journal of Mathematics*, 163:93–124, 2008.

[30] Julian Holstein. Rational Homotopy Theory, master course in Hamburg, course page with lecture notes.

[31] Mark Hovey. *Model Categories*. Number 63. American Mathematical Soc., 1999.

[32] Luc Illusie. *Complexe Cotangent et Déformations I*. Springer, 1971.

[33] John F. Jardine. Lectures on Homotopy Theory, course notes in the page.

[34] John F. Jardine. Presheaves of chain complexes. *K-theory*, 30(4):365–420, 2003.

[35] John F. Jardine. *Local Homotopy Theory*. Springer, 2015.

[36] Jean-Louis Loday. *Cyclic Homology*. Second Edition, Springer-Verlag Berlin Heidelberg, 1998.

[37] Jacob Lurie. Derived Algebraic Geometry, phd thesis, pdf.

[38] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[39] Jacob Lurie. *Higher Algebra*. available in the author's homepage, 2017.

[40] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5. Second Edition, Springer Science & Business Media, 1998.

[41] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag New York Inc., 1992.

[42] Georges Maltsiniotis. Grothendieck ∞-groupoids, and still another definition of ∞-categories, arXiv:1009.2331. 2010.

[43] Marco Manetti. Lectures on deformations of complex manifolds, arXiv:math/0507286. 2005.

[44] Marco Manetti and Francesco Meazzini. Formal deformation theory in left-proper model categories. *New York Journal of Mathematics*, 25:1259–1311, 2019.

[45] James S. Milne. *Étale Cohomology*. Princeton university press, 1980.

[46] Frank Neumann. *Algebraic Stacks and Moduli of Vector Bundles*. IMPA, 2011.

[47] Martin Olsson. *Algebraic Spaces and Stacks*, volume 62. American Mathematical Soc., 2016.

[48] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications Mathématiques de l'IHÉS*, 117:271–328, 2013.

[49] Tony Pantev and Gabriele Vezzosi. Introductory topics in derived algebraic geometry. *Panoramas et Synthèses*, 55(3), 2021.

[50] Jonathan P Pridham. Presenting higher stacks as simplicial schemes, arXiv:0905.4044. *Advances in Mathematics*, 238:184–245, 2013.

[51] Daniel G. Quillen. *Homology of Commutative Rings*. Mimeographed Notes, M.I.T.

[52] Daniel G. Quillen. *Homotopical Algebra*. Springer, 1967.

[53] Daniel G. Quillen. On the (co-)homology of commutative rings. In *Proc. Symp. Pure Math.*, volume 17, pages 65–87, 1970.

[54] Charles Rezk. Toposes and homotopy toposes (version 0.15). 2010.

[55] Emily Riehl. *Categorical Homotopy Theory*, volume 24. Cambridge University Press, 2014.

[56] Emily Riehl. Homotopical categories: from model categories to $(\infty, 1)$-categories, arXiv:1904.00886. 2019.

[57] Joseph J. Rotman. *An Introduction to Homological Algebra*, volume 2. Springer, 2009.

[58] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebraic & Geometric Topology*, 3(1):287–334, 2003.

[59] Michael Shulman. Homotopy limits and colimits and enriched homotopy theory, arXiv:math/0610194. 2009.

[60] Stein Strømme. Elementary introduction to representable functors and Hilbert schemes. *Banach Center Publications*, 36(1):179–198, 1996.

[61] Bertrand Toën. Stacks and non-abelian cohomology, pdf link. 2002.

[62] Bertrand Toën. Higher and derived stacks: a global overview, arXiv:math/0604504. 2006.

[63] Bertrand Toën. Simplicial presheaves and derived algebraic geometry. *Simplicial Methods for Operads and Algebraic Geometry*, pages 119–185, 2010.

[64] Bertrand Toën. Derived algebraic geometry. *EMS Surveys in Mathematical Sciences*, 1(2):153–240, 2014.

[65] Bertrand Toën and Gabriele Vezzosi. From homotopical algebra to homotopical algebraic geometry. In *Lecture at the DFG-Schwerpunkt Workshop, Essen*, 2002.

[66] Bertrand Toën and Gabriele Vezzosi. From hag to dag: derived moduli stacks. In *Axiomatic, Enriched and Motivic Homotopy Theory*, pages 173–216. Springer, 2004.

[67] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry I: Topos theory. *Advances in mathematics*, 193(2):257–372, 2005.

[68] Bertrand Toën and Gabriele Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, volume 2. American Mathematical Soc., 2008.

[69] Bertrand Toën and Gabriele Vezzosi. A note on Chern character, loop spaces and derived algebraic geometry, arXiv:0804.1274. 2008.

[70] Gabriele Vezzosi. Basic structures on derived critical loci. *Differential Geometry and its Applications*, 71, 2020.

[71] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory, homepage. Version of September 2, 2008.

[72] Torsten Wedhorn. *Manifolds, Sheaves, and Cohomology*. Springer, 2016.

[73] David White. Model structures on commutative monoids in general model categories, arXiv:1403.6759. *Journal of Pure and Applied Algebra*, 221(12):3124–3168, 2017.

[74] Ludwig Wittgenstein. *Tractatus Logico-Philosophicus*. translated by D. F. Pears and B. F. McGuinness with an introduction by Bertrand Russel, Revised edition, Routledge & Kegan Paul, 1997.

[75] Alex Youcis. A 'brief' discussion of torsors, in the author's homepage.